Ch-11

Compressible flow

- For compressible flow $\nabla \cdot \vec{v} \neq 0$, Thus mass balance tells us there must be change in fluid density

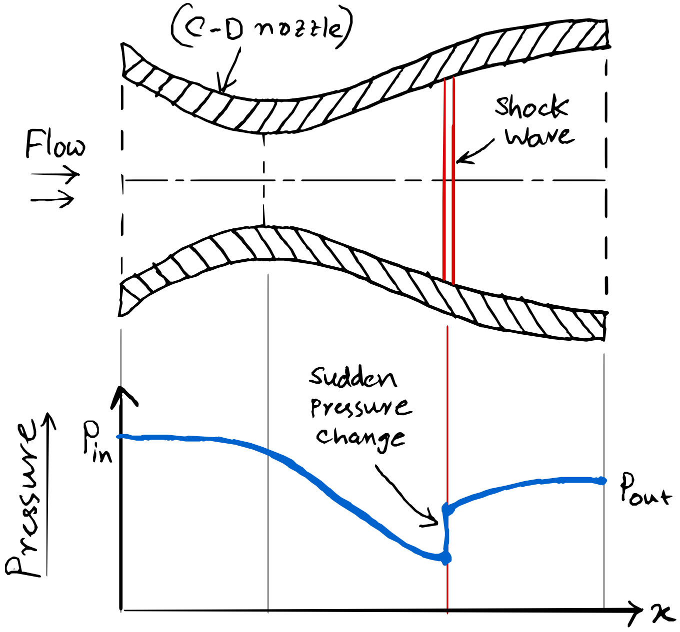
$$\frac{d\rho}{dt} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0$$

- For steady compressible flow,

$$\rho \nabla \cdot \vec{v} + \vec{v} \cdot \underbrace{\nabla \rho}_{\rho = f(\text{space})} = 0$$

- Such change in density is also reflected in pressure change.



* How does pressure and density depends on each other?

(a) Ideal gas model: $PV = nR^*T$

or, $P = \rho R T \longrightarrow R = \left(R^*/M\right)$

(b) Van-der Walls model for non-ideal gases

$$\left(P + \frac{an^2}{V^2}\right)\left(V - nb\right) = nR^*T$$

(Correction in pressure)

(correction in volume)

Ideal Gas Dynamics

* Density : $\rho = \dfrac{P}{RT}$

internal energy (J/kg)

* Internal Energy: $c_v = \left(\dfrac{\partial \tilde{u}}{\partial T}\right)_v = \left(\dfrac{d\tilde{u}}{dT}\right)$

(rate of change in internal energy due to temp change)

(Constant volume process)

→ For state ① to state ② through isochoric process

$$\tilde{u}_2 - \tilde{u}_1 = \int_{T_1}^{T_2} c_v \, dT$$

$$\Rightarrow \tilde{u}_2 - \tilde{u}_1 = c_v \left(T_2 - T_1\right) \qquad \text{if } c_v = \text{const}$$

* Enthalpy: $\tilde{h} = \tilde{u} + \frac{p}{\rho} \longleftarrow$ Pressure energy (flow work)

$$\underset{\substack{\uparrow \\ \text{enthalpy}}}{\tilde{h}} \qquad \underset{\substack{\uparrow \\ \text{internal energy}}}{\tilde{u}}$$

$$C_p = \left(\frac{\partial \tilde{h}}{\partial T}\right)_p = \frac{d\tilde{h}}{dT}$$

→ For state ① to state ② through isobaric process

$$\tilde{h}_2 - \tilde{h}_1 = C_p (T_2 - T_1)$$

* Relation between C_p and C_v:

(a) $\tilde{h} = \tilde{u} + \frac{p}{\rho} = \tilde{u} + RT$ (ideal gas model)

$$\left(\frac{d\tilde{h}}{dT}\right) = \left(\frac{d\tilde{u}}{dT}\right) + R \, , \quad \boxed{C_p = C_v + R}$$

(b) $\boxed{C_p / C_v = K} \longleftarrow$ specific heat ratio

* Combining above two equations give

(a) $C_p = (C_p / K) + R \Rightarrow C_p = \left(\frac{KR}{K-1}\right)$

(b) $C_v = C_p - R \Rightarrow C_v = \left(\frac{R}{K-1}\right)$

* Usually C_p and C_v (as well as K) are temperature dependent. We will assume them constant in this chapter.

* Entropy :

→ Entropy of a sistem does not change if there is no heat transfer and friction involved into the process.

→ The above Sentence just defined a ideal case, where the process is reversible (No friction), and no heat transfer occures to/from the system (adiabatic)

$$\begin{bmatrix}\text{Isentropic} \\ \text{process}\end{bmatrix} = \begin{bmatrix}\text{reversible} \\ \text{process}\end{bmatrix} + \begin{bmatrix}\text{adiabatic} \\ \text{process}\end{bmatrix}$$

* For reversible but <u>not adiabatic</u> process the entropy can change, but the entropy change is reversible.

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta q_{\text{reversible}}}{T}\right)$$

→ The above statement canbe written as

$$ds = \left(\frac{\delta q_{\text{reversible}}}{T}\right)$$

→ For irreversible process (friction is present) the change in entropy is higher than the reversible process.

$$ds \geq \left(\frac{\delta q}{T}\right).$$

* Entropy is a material property, thus related to other properties:

(a) First Tds equation:

$$T ds = d\tilde{u} + p\, d\left(\frac{1}{\rho}\right)$$

(b) Second Tds equation:

$$T ds = d\tilde{h} - \left(\frac{1}{\rho}\right) dP$$

(From thermo-dynamics.)

* Using the definition of $c_v = \frac{d\tilde{u}}{dT}$, the first Tds equation can be written as,

$$T ds = \left(\frac{d\tilde{u}}{dT}\right) dT + \rho R T\, d\left(1/\rho\right)$$

$$\Rightarrow ds = c_v\left(\frac{dT}{T}\right) + \frac{R}{(1/\rho)}\, d\left(1/\rho\right)$$

$$\Rightarrow \boxed{s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{\rho_2}{\rho_1}\right)}$$

* Again using the definition of $c_p = \frac{d\tilde{h}}{dT}$, the second Tds equation can be written as,

$$T ds = \left(\frac{d\tilde{h}}{dT}\right) dT - \left(\frac{RT}{p}\right) dP$$

$$\Rightarrow ds = c_p\left(\frac{dT}{T}\right) - \left(\frac{R}{p}\right) dp$$

$$\Rightarrow \boxed{s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right)}$$

* For isentropic process (Reversible + adiabatic) the net change is entropy is zero.

$$c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right) = 0$$

$$\Rightarrow \left(\frac{R}{K-1}\right)\ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right) = 0$$

$$\Rightarrow \ln\left(\frac{T_2}{T_1}\right)^{1/K-1} = \ln\left(\frac{\rho_2}{\rho_1}\right)$$

$$\Rightarrow \left(\frac{T_2}{T_1}\right)^{K/K-1} = \left(\frac{\rho_2}{\rho_1}\right)^{K}$$

* Similarly

$$c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) = 0$$

$$\Rightarrow \left(\frac{RK}{K-1}\right)\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) = 0$$

$$\Rightarrow \left(\frac{T_2}{T_1}\right)^{K/K-1} = \left(\frac{P_2}{P_1}\right)$$

* Combining above two relations give

$$\frac{P}{\rho^K} = \text{constant.}$$

To remmember from thermodynamics

(a) $c_p - c_v = R$ → always

(b) $c_p/c_v = K$ → always

(c) $S_2 - S_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{\rho_1}{\rho_2}\right)$ → reversible (not adiabatic)

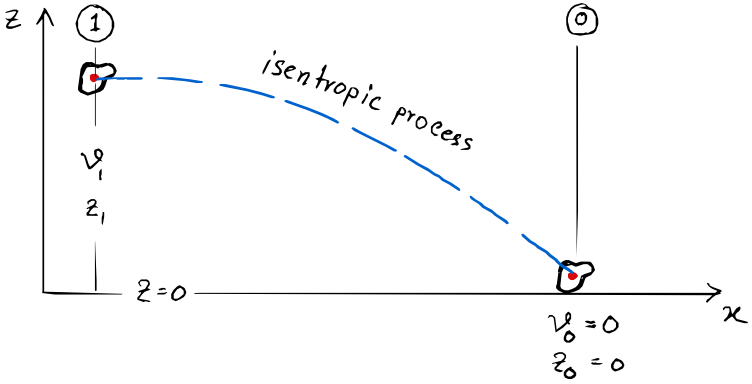
(d) $S_2 - S_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$ → reversible (not adiabatic)

(e) $\left(\frac{T_2}{T_1}\right)^{\frac{K}{K-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{K} = \left(\frac{P_2}{P_1}\right)$ → reversible and adiabatic (isentropic)

(f) $\left(\frac{P}{\rho^K}\right) = \text{constant}$ → reversible and adiabatic (isentropic)

Stagnation Properties

- $T, P, \rho, \tilde{u}, \tilde{h}$ an 's' discussed earlier are all static properties (thermodynamic)
- v and z are other two parameters that contributes to the energy (mechanical)
- Stagnation properties are defined as the properties fluid would obtain if the fluid is bought to $v=0$ and $z=0$ isentropically.



→ Applying energy equation between ① and ②

$$\tilde{h}_1 + \frac{v_1^2}{2} + g z_1 = \tilde{h}_0 + \frac{v_0^2}{2} + g z_0 + \underbrace{q}_{\text{(heat input per kg)}} + \underbrace{W}_{\text{(work input per kg)}}$$

* By definition of stagnation properties,

$v_0 = 0, \; z_0 = 0, \; q = 0, \; w = 0.$

$$\Rightarrow \tilde{h}_0 = \tilde{h}_1 + \frac{v_1^2}{2} + g z_1$$

Specific potential energy

Specific kinetic energy

static enthalpy

Stagnation enthalpy (J/kg)

* If gravity force is negligible

$$\tilde{h}_0 = \tilde{h} + \frac{v^2}{2}$$

* Since $c_p = \frac{d\tilde{h}}{dT} \Rightarrow \tilde{h} = c_p T$ the above expression can be rearranged as,

$$c_p T_0 = c_p T + \frac{v^2}{2}$$

$$\Rightarrow T_0 = T + \frac{v^2}{2c_p} \Rightarrow \boxed{\frac{T_0}{T} = \left(1 + \frac{v^2}{2c_p T}\right)}$$

stagnation temperature (! !)

* Now the expression for stagnation pressure and density can be obtained using isentropic process relationship.

$$\left(\frac{T_0}{T}\right)^{K/K-1} = \left(\frac{\rho_0}{\rho}\right)^{K} = \left(\frac{P_0}{P}\right)$$

which gives,

$$\boxed{\frac{\rho_0}{\rho} = \left(1 + \frac{v^2}{2c_pT}\right)^{1/K-1}}$$

$$\boxed{\frac{P_0}{P} = \left(1 + \frac{v^2}{2c_pT}\right)^{K/K-1}}$$

* Stagnation properties :

(a) $T_0 = T\left(1 + v^2/2c_pT\right)$

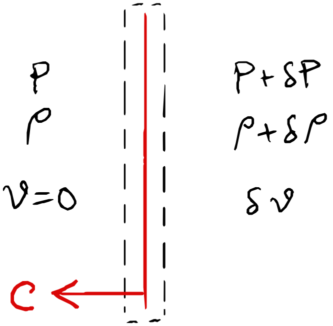
(b) $\rho_0 = \rho\left(1 + v^2/2c_pT\right)^{1/K-1}$

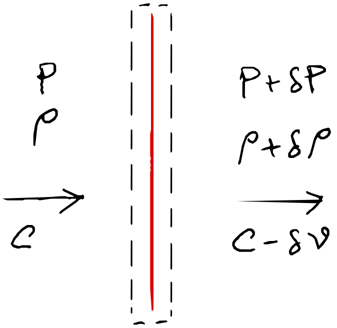
(c) $P_0 = P\left(1 + v^2/2c_pT\right)^{K/K-1}$

Interesting!

Speed of sound

* Similar approach as isolated surface wave.





* mass balance $\int \rho \vec{v} \cdot d\vec{A} = 0$

$$\Rightarrow \quad -\rho A c + (\rho + \delta\rho) A (c - \delta v) = 0$$

$$\Rightarrow \quad \rho c = \rho c - \rho \delta v + c \delta\rho - \underbrace{\underline{\underline{\delta\rho\,\delta v}}}_{\text{small}}$$

$$\Rightarrow \quad \boxed{\rho \delta v = c \delta\rho}$$

* Momentum balance, $\int \vec{v} \rho \vec{v} \cdot d\vec{A} = F \leftarrow \left(\begin{array}{c}\text{Pressure}\\ \times \text{Area}\end{array}\right)$

$$\Rightarrow \quad -c\rho c A + (c - \delta v)\rho(c - \delta v)A = pA - (p + \delta p)A$$

$$\Rightarrow \quad \rho \delta v = \frac{\delta p}{c}$$

(Neglect higher order terms)

* $c^2 = \left(\frac{\delta p}{\delta \rho}\right)$ ⟶ Comes directly from mass & momentum balance!!

$$\Rightarrow \quad c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

↑ isentropic

* Remember bulk modulus,

$$E_v = \left(\frac{dP}{d\rho/\rho}\right) = \rho\left(\frac{\partial P}{\partial \rho}\right)$$

* Thus, $c = \sqrt{\left(\frac{E_v}{\rho}\right)} \longrightarrow$ (Non ideal gas)

* Again, $\frac{P}{\rho^K}$ = Constant (say, m)

Thus, $P = m\rho^K$

$$\Rightarrow \left(\frac{\partial P}{\partial \rho}\right)_s = mK\rho^{K-1} = \frac{P}{\rho^K}\cdot K\rho^{K-1}$$

$$\Rightarrow \left(\frac{\partial P}{\partial \rho}\right)_s = \underbrace{KP/\rho = K(RT)}_{\text{ideal gas}}$$

$$\Rightarrow c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = \sqrt{KRT}$$

ⓐ $c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = \sqrt{\left(\frac{E_v}{\rho}\right)}$ all gasses (Noideal too)

ⓑ $C = \sqrt{KRT}$ (ideal gass only)

Mach number

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* $Ma = \left(\frac{v}{c}\right) = \frac{\text{speed of ??}}{\text{speed of sound}}$

$$\Rightarrow Ma^2 = \left(\frac{v^2}{c^2}\right) = \left(\frac{v^2}{KRT}\right) = \frac{\rho v^2}{PK}$$

Pressure density relation using Ma, Yes !!

* $\left(\frac{v^2}{2C_pT}\right) = ?? \longrightarrow$ Common factor in stagnation properties

$$\left(\frac{v^2}{2C_pT}\right) \xrightarrow{(v=cMa)} \left(\frac{c^2 Ma^2}{2C_pT}\right) \xrightarrow{\left(C_p=\frac{KR}{K-1}\right)} \left(\frac{c^2\cdot Ma^2}{2\cdot\left(\frac{KR}{K-1}\right)T}\right) \longrightarrow \left(\frac{K-1}{2}\right)Ma^2$$

New expression for common factor.

* all stagnation properties can be known from K and Ma.

Stagnation properties and mach number

(a) $\frac{T_0}{T} = \left[1 + \left(\frac{K-1}{2}\right) M_a^2\right]$

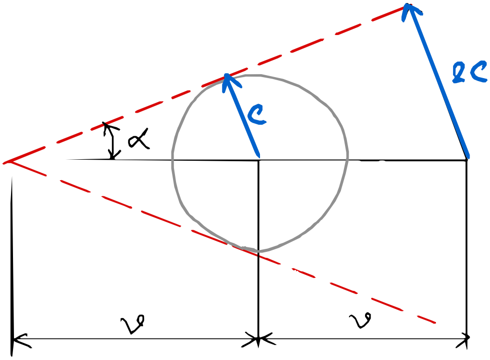
(b) $\frac{\rho_0}{\rho} = \left[1 + \left(\frac{K-1}{2}\right) M_a^2\right]^{1/K-1}$

(c) $\frac{P_0}{P} = \left[1 + \left(\frac{K-1}{2}\right) M_a^2\right]^{K/K-1}$

Mach Cone (Ma > 1)

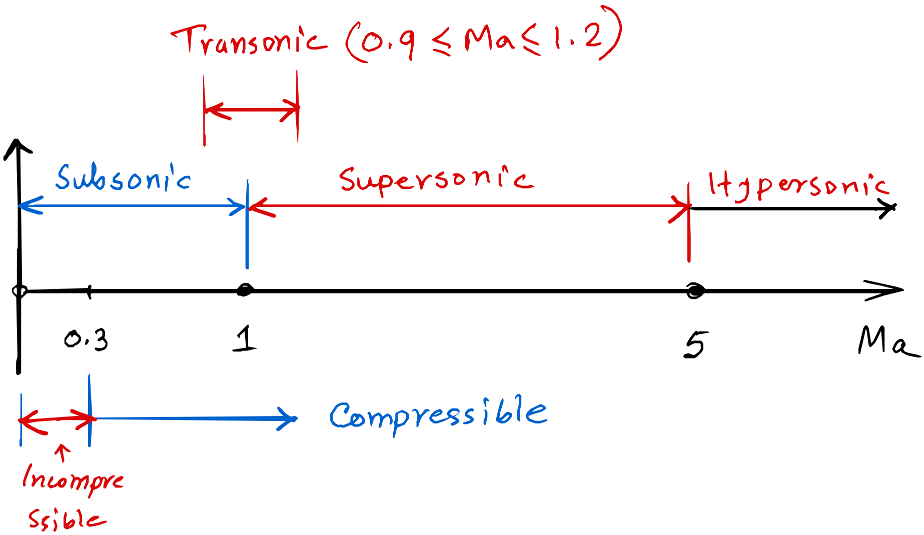
** See video on mach cone.

→ How v and c are related in mach cone formation.



$$\Rightarrow \sin\alpha = \left(\frac{c}{v}\right) = \left(\frac{1}{M_a}\right)$$ (Simple!!)

Flow regimes on Ma line

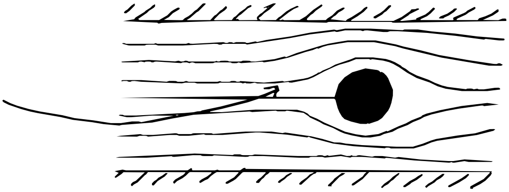


Open channel flow	Compressible flow
ⓐ Gravity driven flow	ⓐ Inertia driven flow
ⓑ $Fr = \frac{\text{inertia force}}{\text{gravitational force}}$	ⓑ $Ma = \frac{\text{inertia force}}{\text{compressibility force}}$
ⓒ $Fr = \frac{v}{\sqrt{gy}} = \frac{v}{v_c}$	ⓒ $Ma = \frac{v}{\sqrt{kRT}} = \frac{v_c}{c}$
ⓓ Critical condition $Fr = 1$	ⓓ Sonic condition $Ma = 1$

* Fluid element comunication:

(a) Ma < 1, * Sound (Pressure wave) can travel upstream.

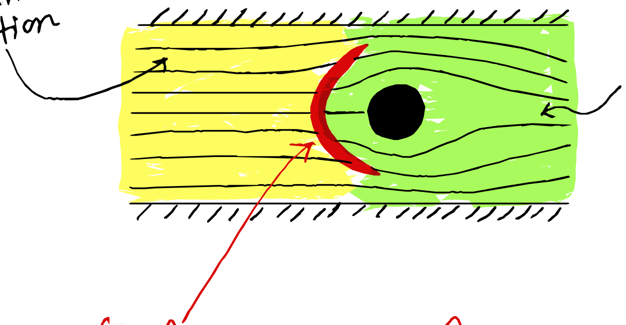
* Upcoming fluids are informed well ahead of any obtacle



fluids decelerates far ahead of the obstacle

(b) Ma > 1, * Sound (pressure wave) can not travel upstream.

* Upcoming fluids are unaware about any obstacle ahead.



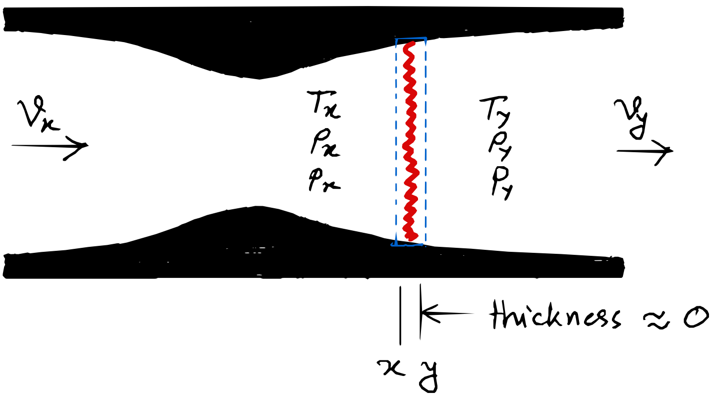
No upstream comunication (Ma > 1)

upstream comuni-cation (Ma < 1)

fluids pile up and changes density

Normal shock wave

* shock wave can move ??



* Mass balance, $\int \rho \vec{v} \cdot d\vec{A} = 0$

$$\Rightarrow \rho_x v_x A_x = \rho_y v_y A_y$$

$$\left(\rho_x = \frac{P_x}{RT_x}\right)$$

$$\left(A_x = A_y\right)$$

$$\begin{pmatrix} v_x = Ma_x\, c_x \\ v_x = Ma_x \sqrt{kRT_x} \end{pmatrix}$$

* Substitution
\+
rearrangement

$$\boxed{\frac{Ma_y}{Ma_x} = \left(\frac{P_x}{P_y}\right)\sqrt{\frac{T_y}{T_x}}}$$

* Simply mass balance

* Momentum balance, $\int \vec{v} \rho \vec{v} \cdot d\vec{A} = \Sigma \vec{F}$

$$\Rightarrow P_x A - P_y A = \rho_y v_y^2 A - \rho_x v_x^2 A \qquad (A_x = A_y)$$

$$\Rightarrow P_x - \rho_x v_x^2 = P_y + \rho_y v_y^2$$

* Need P, ρ relation in terms of Ma.

$$P = \left(\frac{\rho v^2}{k\, Ma^2}\right)$$

$$\Rightarrow \boxed{\left(\frac{P_y}{P_x}\right) = \left(\frac{1 + k\, Ma_x^2}{1 + k\, Ma_y^2}\right)}$$

* Simply momentum balance

* Energy balance,

$$\begin{bmatrix}\text{Total energy} \\ \text{@ } x \text{ side}\end{bmatrix} = \begin{bmatrix}\text{Total energy} \\ \text{@ } y \text{ side}\end{bmatrix}$$

* No Q_{in} & W_{in}

$$\Rightarrow \tilde{h}_{o,x} = \tilde{h}_{o,y} \longrightarrow T_{o,x} = T_{o,y}$$

$$c_{p,x} = c_{p,y}$$

$$\Rightarrow T_{o,x} = T_{o,y}$$

$$\Rightarrow T_x \left(1 + \frac{k-1}{2} Ma_x^2\right) = T_y \left(1 + \frac{k-1}{2} Ma_y^2\right)$$

* Rearrangement gives, $\frac{T_y}{T_x} = \frac{1 + (k-1)/2 \, Ma_x^2}{1 + (k-1)/2 \, Ma_y^2}$

Summary

Mass balance → $\frac{T_y}{T_x} = \left(\frac{Ma_y}{Ma_x}\right)^2 \left(\frac{P_y}{P_x}\right)^2$

Momentum balance → $\frac{P_y}{P_x} = \left(\frac{1 + k Ma_x^2}{1 + k Ma_y^2}\right)$

Energy lalance → $\frac{T_y}{T_x} = \left(\frac{1 + (k-1)/2 \, Ma_x^2}{1 + (k-1)/2 \, Ma_y^2}\right)$

* Our target is to obtain the possible relations between upstream and down stream Mach number that satifies both mass, moment and energy conservation.

* One trivial solution to above set of equations is, $Ma_y^2 = Ma_x^2$, which is true for case with no shock wave.

* Substitution of paramutors reveals that the relationship between Ma_x & Ma_y is a quadratic relationship (2 solutions)

* Thus beside $Ma_x^2 = Ma_y^2$, there exist another solution:

$$Ma_y^2 = \left\{ \frac{Ma_x^2 + \left(\frac{2}{K-1}\right)}{\left(\frac{2K}{K-1}\right) Ma_x^2 - 1} \right\}$$

* What would curve of Ma_y Vs Ma_x plot look like?

→ For $K=1.5$, (Example)

$$Ma_y^2 = \left(\frac{Ma_x^2 + 4}{6\,Ma_x^2 - 1} \right)$$

→ Ma_x and Ma_y always lies on other side $(K>1)$.

Variable area duct (1-D)

* From $\vec{F} = m\vec{a}$ along streamline:

$$dp + \frac{1}{2}\rho\, d(v^2) + \gamma\, \cancel{dz} = 0 \qquad \text{(Chapter-3)}$$

{For gas & light weight fluids}

$$\Rightarrow dp = -\frac{1}{\cancel{2}}\rho\, \cancel{2} v\, dv$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{fractional} \\ \text{pressure} \\ \text{change}}}{\left(\frac{dp}{\rho v^2}\right)} = -\underset{\substack{\uparrow \\ \text{franctional} \\ \text{velocity} \\ \text{change}}}{\left(\frac{dv}{v}\right)} \quad \text{-------- ①}$$

* Mass balance, $\rho A v =$ constant $(\dot{m} = \rho A v)$

(applicable for compressible fluids)

$$\Rightarrow \ln\rho + \ln A + \ln v = \text{constant}$$

$$\Rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} = 0 \quad \text{------ (II)}$$

* Combining two expressions

$$\frac{dp}{\rho v^2} = \underset{\longrightarrow \text{ Pressure dependent term}}{\frac{d\rho}{\rho}} + \frac{dA}{A} \longrightarrow \text{Pressure independent term}$$

* Take the pressure independent term in one side,

$$\frac{dP}{\rho v^2}\left[1-\frac{v^2}{dP/d\rho}\right] = \frac{dA}{A}$$

* We already know, $c = \sqrt{\left(\frac{dP}{d\rho}\right)_s}$

$$\Rightarrow \frac{dP}{\rho v^2}\left[1-Ma^2\right] = \left(\frac{dA}{A}\right)$$

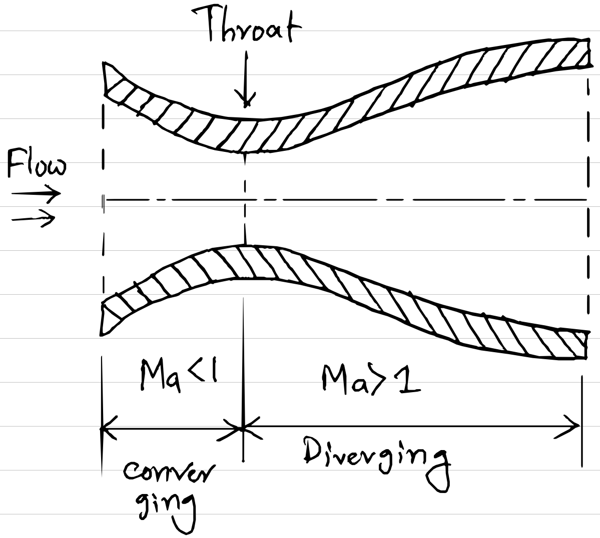
$$\Rightarrow \underbrace{\left(\frac{dv}{v}\right)}_{\text{Fractional change in velocity}} = -\underbrace{\left(\frac{dA}{A}\right)}_{\text{Fractional change in area}}\left(\frac{1}{1-Ma^2}\right) \quad \left(\text{from } ①\right)$$

* For $Ma < 1$: $\underbrace{\left(\frac{dv}{v}\right)}_{\text{decreases velocity} \downarrow} \propto -\underbrace{\left(\frac{dA}{A}\right)}_{\text{increasing area} \uparrow}$

* For $Ma > 1$: $\underbrace{\left(\frac{dv}{v}\right)}_{\text{Increases velocity} \uparrow} \propto +\underbrace{\left(\frac{dA}{A}\right)}_{\text{increasing area} \uparrow}$

Supersonic Nozzle

1) Also called "converging-diverging nozzle"
(C-D Nozzle)



* Velocity / Ma variation