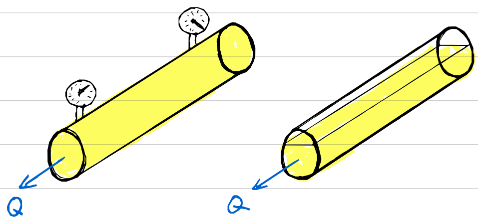
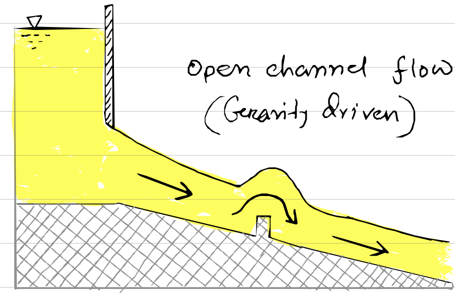
Ch-10

Open channel flow

* Gravity driven flow (Not pressure driven)



Pipe flow (Pressure driven) | Open channel flow (Gravity driven)



Open-channel flow

- uniform flow (depth = const)
- non-uniform flow (depth ≠ constant)
- laminar ($Re < 500$) (open channel flows are rarely laminar)
- turbulent ($Re > 12 \times 10^3$) (Most common in open channel flow problems)

* We will limit our study to homogeneous open-channel flow (No stratified flows)

* Since such flows are gravity driven, Froude Number is very important. Based on Fr open channel flows can be classified in 3 categories
  (a) Supercritical flow $(Fr > 1)$ "rapid flow"
  (b) Critical flow $(Fr = 1)$
  (c) Subcritical flow $(Fr < 1)$ "tranquil flow"

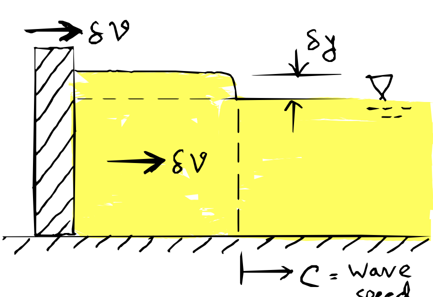
* Froude Number:

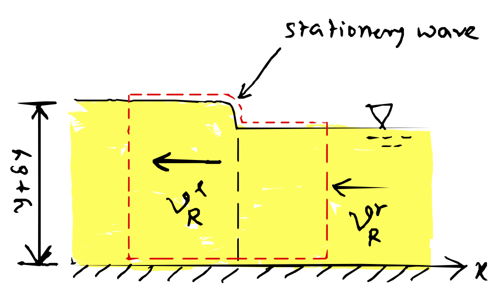
$$Fr = \left(\frac{v}{\sqrt{gl}}\right) = \left(\frac{\text{inertia}}{\text{gravity}}\right)$$

Solitary surface wave

* For open channel flow the free surface can deform (unique).

* Consider a case where the free surface is deformed by moving a wall at speed δV.





* To a stationary observer the flow is unsteady and the wave moves at speed c,

→ For such observer there will be no fluid motion ahead of the wave.

* To an observer moving at speed c the flow is stationary. To him the flow at his/her left will be $\vec{v}_R^{\,l} = (-c + \delta v)\hat{i}$ and at right $\vec{v}_R^{\,r} = (-c)\hat{i}$.

* Continuity equation (mass balance). $\int_{cs} \vec{v} \cdot d\vec{A} = 0$

$$-cyb = (-c + \delta v)(y + \delta y)b \qquad (b \text{ is into page})$$

$$\Rightarrow c = \frac{(y + \delta y)\delta v}{\delta y} \qquad \Rightarrow \underline{cy = (c - \delta v)(y + \delta y)}$$

Use later in momentum Eq

* For small amplitude wave, $\delta y \ll y$, the above expression becomes,

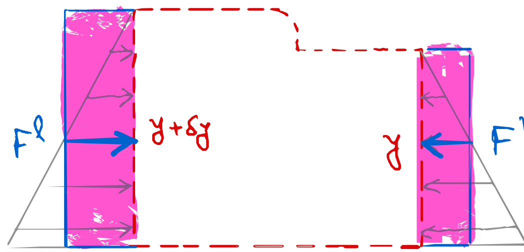
$$c = y\left(\frac{\delta v}{\delta y}\right) \qquad \text{———} \qquad (1)$$

* Now momentum equation can be applied as

$$\int_{cs} \vec{v} \rho \vec{v} \cdot d\vec{A} = \Sigma \vec{F}$$

* We consider the x-component of the above equation.

* Now, $\sum F_x$ can be obtained from the hydrostatic force acting on the left and right control surfaces.



$$* \quad \sum F_x = F^l - F^r = \rho g \left(\frac{y+\delta y}{2}\right)(y+\delta y)\, b - \rho g \left(\frac{y}{2}\right) y b$$

$$= \frac{1}{2}\rho b g\left[(y+\delta y)^2 - y^2\right]$$

* Again $\int_{cs} \vec{v}\rho\,\vec{v}\cdot d\vec{A}$ x component can be calculated as, $\int_l \vec{v}\rho\,\vec{v}\cdot d\vec{A} + \int_r \vec{v}\rho\,\vec{v}\cdot d\vec{A}$

* For left control surface, $\int_l \vec{v}\rho\,\vec{v}\cdot d\vec{A} = -v_R^l\,\rho\,v_R^l A_l$

$$\Rightarrow \int_l \vec{v}\rho\,\vec{v}\cdot d\vec{A} = -(-c+\delta v)\,\rho\,(-c+\delta v)\,(y+\delta y)\,b$$

$$= -\rho b\,(y+\delta y)\,(c-\delta v)^2$$

$$= -\rho b c y\,(c-\delta v)$$

$\boxed{\text{use mass balance } cy = (c-\delta v)(y+\delta y)}$

* For right control surface, $\int_r \vec{v}\rho\,\vec{v}\cdot d\vec{A} = (-c)\rho(-c)yb$

$$\Rightarrow \int_r \vec{v}\rho\,\vec{v}\cdot d\vec{A} = \rho b c^2 y$$

* Combining all together we get,

$$-\rho b c y(c-\delta v)+\rho b c^2 y=\frac{1}{2}\rho b g\left[(y+\delta y)^2-y^2\right]$$

$$\Rightarrow -\rho b c y\,(\cancel{c}-\delta v+\cancel{c})=\frac{1}{2}\rho b g\left[\cancel{y^2}+2y\delta y+\delta y^2-\cancel{y^2}\right]$$

$$\Rightarrow \rho b c y\,\delta v=\rho b g y\,\delta y+\frac{1}{2}\rho b g\,\delta y^2$$

* For small amplitude wave, $\delta y \ll y$ or $\delta y^2 \ll y\delta y$, the above expression reduces to,

$$(\rho b c y)\,\delta v=(\rho b g y)\,\delta y$$

$$\Rightarrow \left(\frac{\delta v}{\delta y}\right)=\left(\frac{g}{c}\right)$$

* From (1) we get, $c=y\dfrac{\delta v}{\delta y} \Rightarrow c=y\,g/c$

$$\Rightarrow c=\sqrt{gy}$$ (this is interesting)

Comments :

(i) c is independent of δy !!

(ii) c is independent of ρ ? why ?

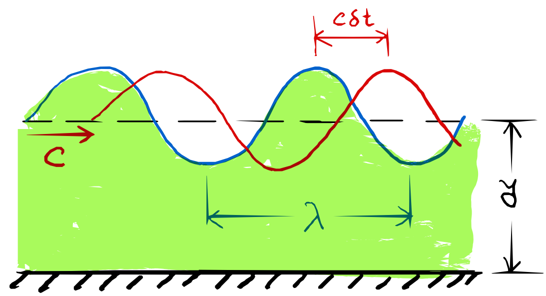
→ fluid inertia $\propto \rho c$
→ hydrostatic force $\propto \rho g$
} ratio matters only.

(Think !!)

* For sinusoidal surface wave (more tricky calculation required), the wave speed is found as,

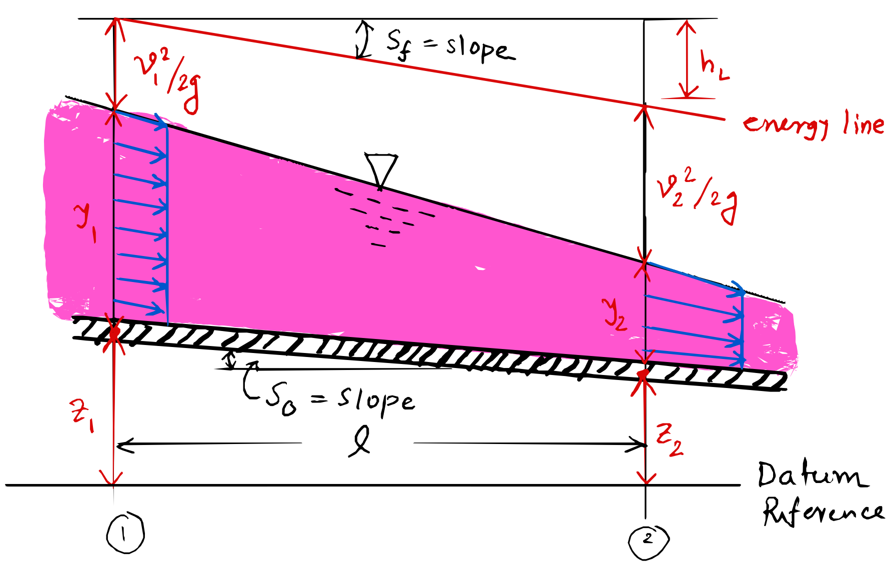
$$c = \sqrt{\frac{g\lambda}{2\pi}}$$

(valid for $\lambda \ll y$)



Energy consideration

* The energy equation (Bernoulli's equation) can be applicable since most of the flow is inviscid.
→ we can account for the frictional loss



* Energy equation between section ① and ②

$$\underbrace{\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + z_1}_{\text{Total energy at section ①}} = \underbrace{\frac{P_2}{\rho g} + \frac{v_2^2}{2g} + z_2}_{\text{Total energy at section ②}} + \underbrace{h_L}_{\text{energy loss between section ① \& ②}}$$

* Here, $P_1/\rho g = y_1$ and $P_2/\rho g = y_2$. Thus

$$y_1 + \left(\frac{v_1^2}{2g}\right) + (z_1 - z_2) = y_2 + \frac{v_2^2}{2g} + h_L$$

* The <u>bottom slope</u> $S_0 = \left(\frac{z_1 - z_2}{l}\right) \Rightarrow (z_1 - z_2) = S_0 l$

$$\Rightarrow y_1 + \frac{v_1^2}{2g} + S_0 l = y_2 + \frac{v_2^2}{2g} + h_L$$

* Now the slope of the energy line represents loss due to friction (h_L) per unit length of the channel, $S_f = (h_L/l)$. S_f = <u>friction slope</u>

$$\Rightarrow y_1 + \left(\frac{v_1^2}{2g}\right) + S_0 l = y_2 + \left(\frac{v_2^2}{2g}\right) + S_f l$$

$$\Rightarrow \underbrace{y_1 + \left(\frac{v_1^2}{2g}\right)}_{\text{Specific energy at section ①}} = \underbrace{y_2 + \left(\frac{v_2^2}{2g}\right)}_{\text{specific energy at section ②}} + (S_f - S_0) l$$

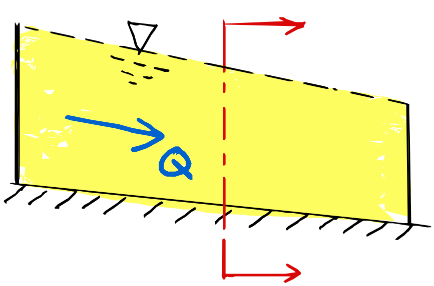
* Assuming $E_1 = y_1 + \left(\frac{v_1^2}{2g}\right)$ and $E_2 = y_2 + \left(\frac{v_2^2}{2g}\right)$ we obtain the relation between the specific energies at section ① and ②,

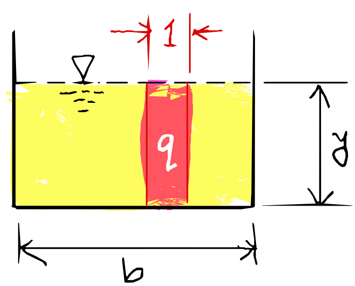
$$E_1 = E_2 + (S_f - S_o)\,l$$

* Remember, specific energy is simple the sumation of pressure energy and kinetic energy, expressed as head.

Rectangular channel flow

* The energy equation $E_1 = E_2 + (S_f - S_o)\,l$ is valid for any cross-section of the channel. Here we consider a flow through rectangular channel of width b (into page)





* The total flow rate Q (m^3/s) can be expressed as flow rate per unit depth q (m^2/s) as

$$q = Q/b$$

* Velocity can be expressed as $v = \left(\frac{q}{y}\right)$.

Substitution of v expression (interms of q) in to the definition of specific energy gives

$$E = y + \frac{q^2}{2gy^2} \qquad \text{—————— } (2)$$

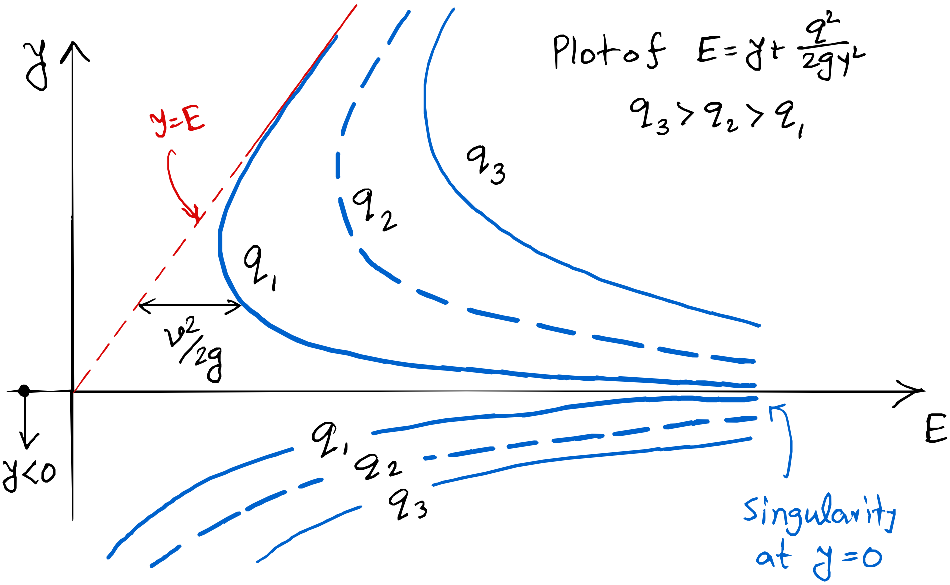
* The above equation can be rearranged as,

$$y^3 - (E)\,y^2 + \left(q^2/2g\right) = 0 \qquad \text{———— } (3)$$

* The above equation reveals that for a given flow rate (per depth) q, there can be three (03) different flow depth !!

* How would y Vs E plot look like?

(See next figure)



* We can see:

→ there is a sigularity at $y=0$.

→ The plot extend in to region $y<0$.
(No physical significance)

→ The maximum slope obtainable is 1
(See line: $y=E$)

→ There is a minimum value of E for any given flow rate q.

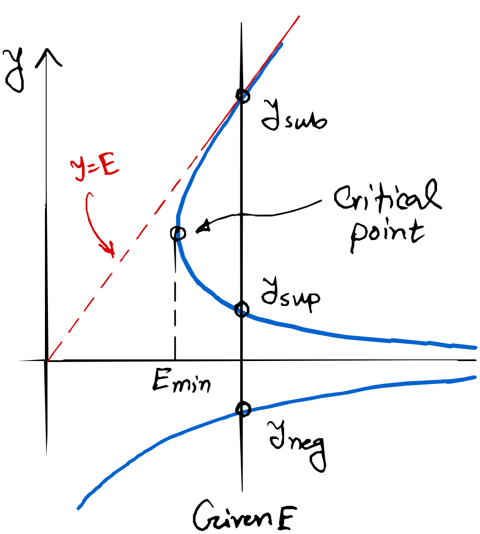
→ look how $v^2/2g$ varies with E and y. (Explain).

* Since Eq-3 is a 3rd order equation it should have 3 solutions (y value) for a given flow rate
  - → Two of them are possitive (+ve)
  - → One of them are negative (−ve)

* The negative solution ($y<0$) has no physical significance. But the two positive y values for given flow rate are known alternate depth (Same flow rate and same specific energy).

* The minimum E point is called critical point.

* Find the location (y_c, E_c) of the critical point for a given flow rate q.



* How to get minima of a function?

$$dE/dy = 0$$

* Setting $dE/dy = 0$ gives,

$$\frac{dE}{dy} = 1 - 2\frac{q^2}{2gy_c^3} = 0, \Rightarrow y_c = \left(\frac{q^2}{g}\right)^{1/3}$$

* To obtain $E_c = E_{min}$, we sett $y = y_c$ in equation (2):

$$E_c = E_{min} = y_c + \frac{q^2}{2g\,y_c^2} = y_c + \frac{1}{2}y_c = \left(\frac{3y_c}{2}\right)$$

$$\Rightarrow E_c = E_{min} = \frac{3}{2}\left(\frac{q^2}{g}\right)^{1/3}$$

* From these expressions we can calculate the critical velocity as,

$$E_c = y_c + \frac{v_c^2}{2g}$$

$$\Rightarrow v_c^2 = 2g(E_c - y_c) = 2g\left[\frac{3}{2}y_c - y_c\right]$$

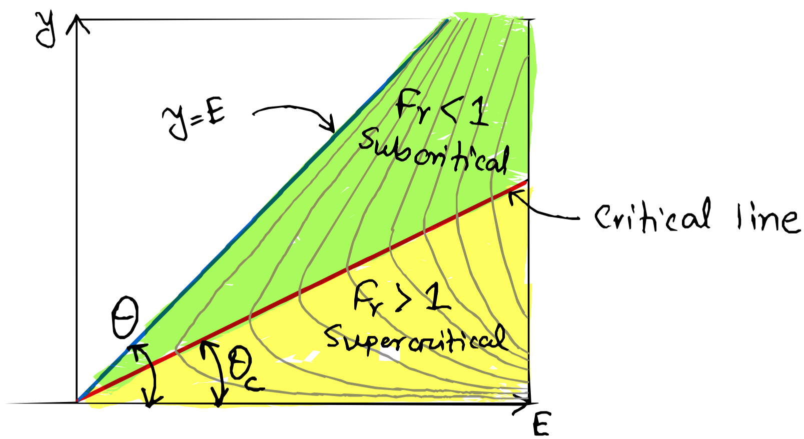
$$\Rightarrow v_c = \sqrt{g\,y_c} = (qg)^{1/3}$$

* What is the Froude number at critical point?

$$Fr_c = \frac{v_c}{\sqrt{g\,y_c}} = 1$$

* Thus, the entire flow regime can be devided in to 3 caragories:

1. critical flow ($F_r = 1$) * gravity ≈ inertia
2. Subcritical flow ($F_r < 1$) * gravity dominates
3. Supercritical flow ($F_r > 1$) * inertia dominates



Slope of the critical line

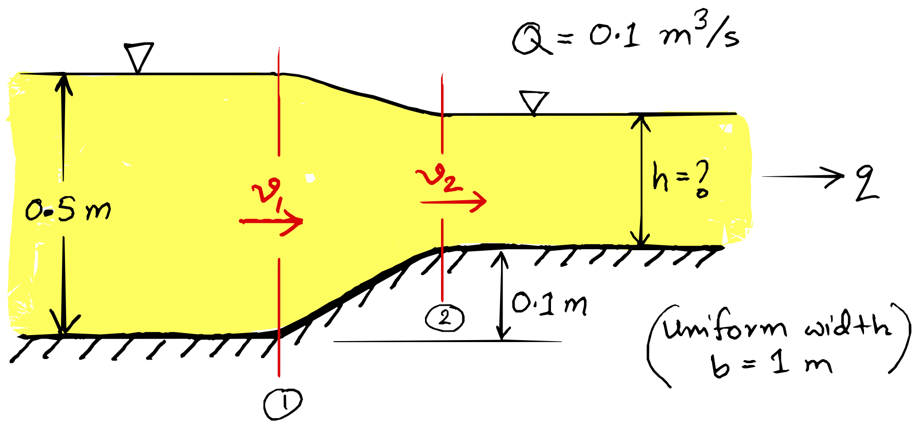
* To obtain the equation for critical line we need to express $E_c = f(y_c)$ or $y_c = f(E_c)$

We have, $E_c = \left(\frac{3}{2}\right) y_c \longrightarrow$ straight line through (0,0).

* Slope of critical line,

$$\frac{dy_c}{dE_c} = \left(\frac{2}{3}\right) \longrightarrow \theta_c = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.7^\circ$$

* Example problem:



* All changes happen between section ① and ②.

Since there is no frictional loss ($h_L = 0$)

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2$$

Here, $y_1 = 0.5\,m$, $v_1 = \dfrac{Q}{y_1 b} = 0.2\ m/s$, $z_1 = 0$

$v_2 = ?$, $y_2 = h = ?$, $z_2 = 0.1\,m$

$$\Rightarrow\ 0.5 + \left(\frac{0.2^2}{2g}\right) + 0 = h + \left(\frac{v_2^2}{2g}\right) + 0.1$$

$$\Rightarrow\ 0.502 = h + \frac{v_2^2}{2g} + 0.1$$

$$\Rightarrow\ h + \frac{0.1^2}{2g\,y_2^2} = 0.402$$

$$\Rightarrow\ h + 5.102\times10^{-4}/h^2 = 0.402$$

$$\Rightarrow\ h^3 - 0.402\,h^2 + 5.102\times10^{-4} = 0$$

mass balance

$v_1 y_1 b = v_2 h b$

$\Rightarrow v_2 = (v_1 y_1)/h$

$\Rightarrow v_2 = 0.1/h$

* Solving for h (calculator)

$h_1 = 0.3988\ m$, $h_2 = 0.0374\ m$, $h_3 = -0.0342\ m$
(unphysical)

* Thus, $h = 0.3988\ m$ or $0.0374\ m$

which one ??

* Specific energy has the answer !!!

* Since there is no loss the specific energy must be conserved,

$$E_1 = E_2 + (S_f - S_o)\,l \longrightarrow S_f = \left(\frac{h_L}{l}\right) = 0$$

$$\Rightarrow E_1 = E_2 - S_o \cdot l \longrightarrow S_o = \left(\frac{z_1 - z_2}{l}\right)$$

$$\Rightarrow E_1 = E_2 - (z_1 - z_2)$$

$$\Rightarrow E_1 - E_2 = 0.1 \quad \text{————} (*)$$

* which value of h (h_1 or h_2) satisfy the above condition ?

For h_1 :
$$\begin{cases} E_1 = y_1 + \left(\frac{v_1^2}{2g}\right) = 0.502 \\ E_2 = h + \left(\frac{v_2^2}{2g}\right) = h + \frac{(v_1 y_1)^2}{2g\,h^2} = 0.402 \end{cases}$$

$\therefore E_1 - E_2 = 0.1$ (satisfied)

For h_1 :
$$\begin{cases} E_1 = y_1 + \left(\frac{v_1^2}{2g}\right) = 0.502 \\ E_2 = h + \left(\frac{v_2^2}{2g}\right) = h + \frac{(v_1 y_1)^2}{2g\,h^2} = 0.402 \end{cases}$$

$\therefore E_1 - E_2 = 0.1$ (satisfied) !!! what ??

* This is expected as h_1 and h_2 are both obtained from energy conservation. Now what to do ?

→ Energy is conserved,
→ what about mass ?

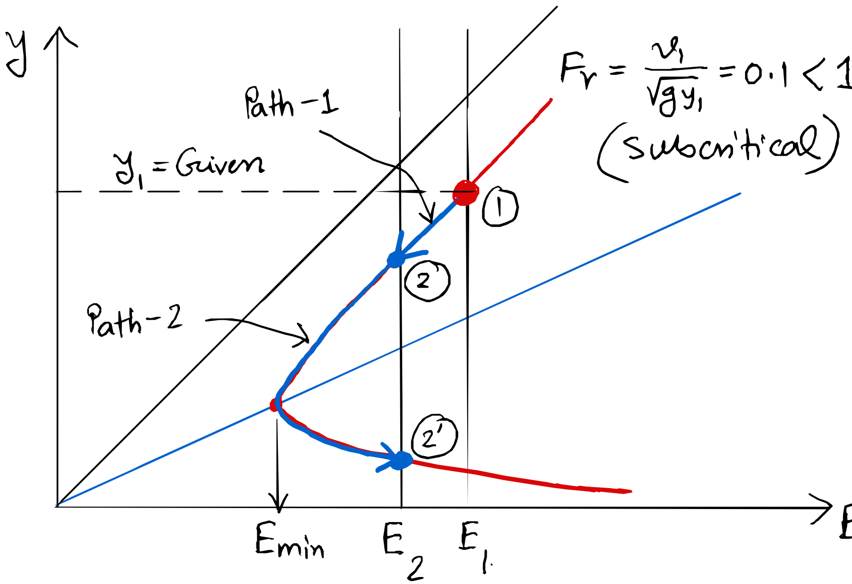
inflow, $y_1 b v_1 = (0.5 \times b \times 0.2) = 0.1b\ m^3/s$

outflow, $h b v_2 = h b (y_1 v_1)/h = y_1 b v_1$
$= (0.5 \times b \times 0.2) = 0.1b\ m^3/$

⇒ automatically satisfied. !!!

* what other physical constraint(s) the flow must satisfy ??

* Mass must not change at any point during flow transformation (all points must be on a single q line)



* Path-1: speific energy decreases gradually

$$E_1 \longrightarrow E_2$$

* Path-2: speific energy decreases to E_{min} and then increases.

$$E_1 \searrow E_{min} \nearrow E_2$$

* How can we change speific energy?

$$E_{total} = \left(P + \frac{h^2}{2g}\right) + z$$

$$\Rightarrow \underline{\underline{E_{total}}} = E + Z$$

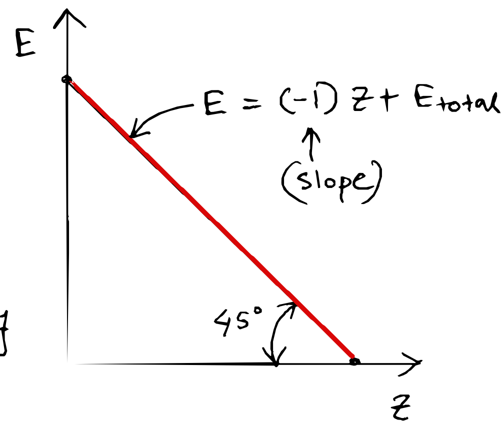
Constant

only way to change E is to change z.

* $E = E_{total} - z$

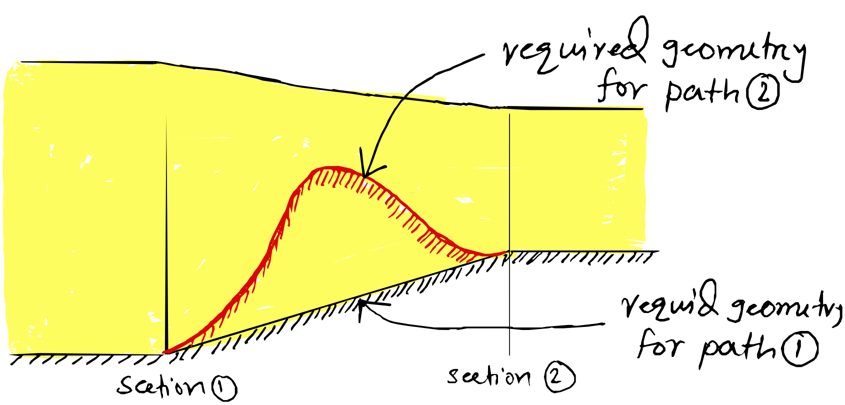
→ E decrease linearly if z is increased linearly.

→ E increases linearly if z is decreased linearly.



* For path ①: For gradual decrease in E_1 to E_2, z must be gradually increased from section ① to section ② ✓ (OK)

* For path ②: For gradual decrease in E_1 to Emin, z must be gradually increased and for gradual decrease in Emin to E_2, z must be decreased gradually.



* Since, the given geometry allows flow to remain subcritical, the physically accessible path is path -1.

* Due to the channel geometry the flow is not allowed to access supercritical regime. This phenomena is often refered as "flow accessibility".

* Still need to select which h ?

→ It is certain that flow depth must decrease with $F_r < 1$.

* For $h_1 = 0.3988$ m $[< y_1 \, (0.5\,m)]$

$$(F_r)_{(2)} = \left(\frac{v_2}{\sqrt{g h_1}}\right) = \frac{v_1 y_1}{h_1 \sqrt{g h_1}} = 0.127 \; (<1) \qquad (OK)$$

* For $h_2 = 0.0374$ m $[< y_1 \, (0.5\,m)]$

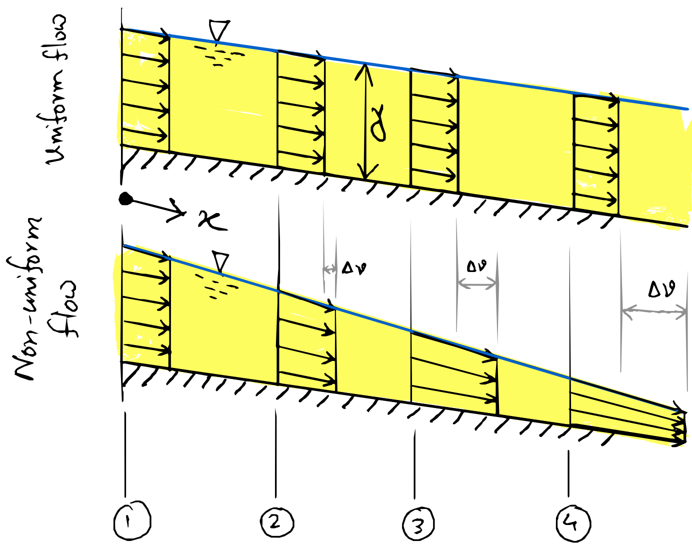
$$(F_r)_{(2)} = \left(\frac{v_2}{\sqrt{g h_2}}\right) = \frac{v_1 y_1}{h_1 \sqrt{g h_1}} = 4.41 \; (>1) \quad (\text{Not OK})$$

* Thus the correct answer is $h = 0.3988$ m

(Ans)

Uniform open channel Flow

* Flow depth and velocity does not changes along channel length.



* Thus we can define uniform flow as $\frac{dy}{dx} = 0$

  → Analogous to fully developed flow.

* Can we convert any non uniform flow into a uniform flow by changing the bottom slope (S_0)?

  → what about $S_0 = 0$?

  * No flow is possible for $S_0 = 0$.

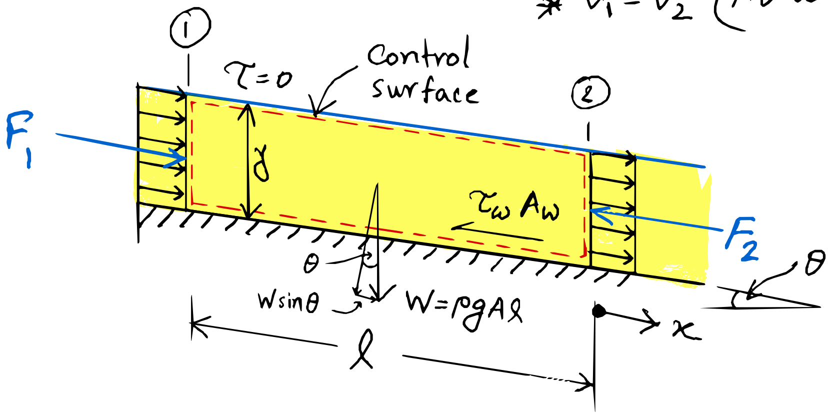
  → what about $S_0 = S_f$?

  * Can have flow without loss !!

Chezy Equation

* what is the shear stress at the free surface?
  (Always remember $\tau_{free, surface} = 0$)

* $v_1 = V_2$ (No acceleration)



* What is the weight contained in the channel between section ① and ② that are separated by distance l?

$$W = (\rho g) A l \quad (A = \text{bottom area})$$

* Force balance at the control volume gives

$$\Rightarrow F_1 - F_2 - \tau_w A_w + W \sin\theta = 0$$

$$\Rightarrow F_1 - F_2 - \tau_w P_w l + \rho g A l = 0$$

(wetted perimeter)

* what are F_1 and F_2 ?
  - → Hydrostatic force
  - → Same at both section $(y_2 = y_1)$

Thus, $\tau_w P_w l = \rho g A l \sin\theta$

* $l \sin\theta$ can be replaced by bottom slope

$$S_0 = \left(\frac{z_1 - z_2}{l\cos\theta}\right) = \frac{l\sin\theta}{l\cos\theta} = \tan\theta \approx \sin\theta$$

(Small θ)

* Replacement of $\sin\theta = S_0$ gives,

$$\tau_w P_w l = \rho g A l S_0$$

$$\Rightarrow \tau_w = \left(\frac{\rho g A S_0}{P_w}\right) = \rho g R_h S_0$$

Here, $R_h = A/P_w$, called hydraulic radius.

* For turbulant flow (most open channel flows) we know that the friction factor is not dependent on viscosity, rather on dynamic pressure.

$$\tau_w = (K)\frac{1}{2}\rho v^2 \quad \text{(wholly turbulant)}$$

* Thus, $(K)\frac{1}{2}\rho v^2 = \rho g R_h S_0$

$$\Rightarrow v^2 = \underbrace{\left(\frac{2\rho g}{\rho K}\right)}_{\text{constant}} R_h S_0$$

$$\Rightarrow \boxed{v = (\text{Const})\sqrt{R_h S_0}}$$

Velocity = f(only geometry)

(The const. must be determined experimentally)

* This equation is known as chezy equation developed by French engineer chezy (1768).

* Experimentally observed that

$v \propto \sqrt{S_0}$ (OK)

$v \propto R_h^{2/3}$ (Not $v \propto R_h^{1/2}$)

Thus, proper representation should look like,

$$v = (\text{Const})\, R_h^{2/3} \cdot S_0^{1/2}$$

Manning equation

* Manning modified the Chezy equation (1889) with consideration of surface conditions.

$$v = \left(\frac{R_h^{2/3}}{n}\right) S_o^{1/2}$$

→ n is experimentally obtained.

→ n is known as Manning Resistance coefficient (or Manning's "n").

* See table 10.1 from textbook (Page 573).

* Note: * Manning's "n" is not dimensionless.

    * It has unit of $\left(\frac{s}{m^{1/3}}\right)$.

    ** Table 10.1 must be used in SI-system of unit. Otherwise use conversion factor.

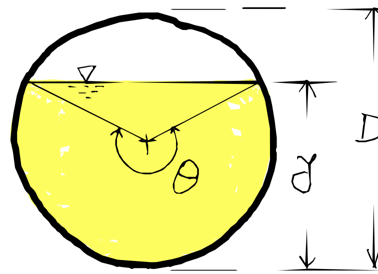
* From velocity flowrate can be obtained as,

$$Q = vA = A\left(\frac{R_h^{2/3}}{n}\right) S_o^{1/2}$$

* Maximum flow rate

→ Not a pipe flow.
So must be partially filled ($0 \le y < D$)

→ To create flow we must have non zero bottom slope S_o.



* Manning equation, $Q = A\left(\frac{R_h^{2/3}}{n}\right) S_o^{1/2}$

→ Area A can be shown as, $A = \left(\frac{D^2}{8}\right)(\theta - \sin\theta)$

→ Wetted perimeter, $P = \left(\frac{D\theta}{2}\right)$

→ Hydraulic radius, $R_h = \left(A/P\right) = \frac{D(\theta - \sin\theta)}{4\theta}$

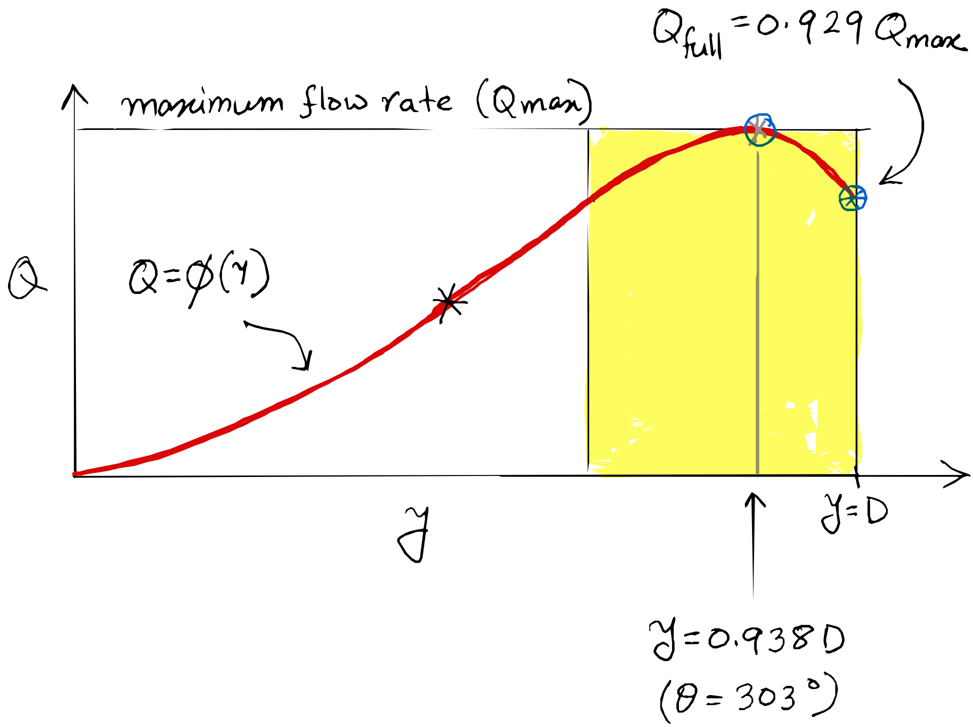
* Flow rate, $Q = \frac{D^2(\theta - \sin\theta)}{8n} \times \frac{D^{2/3}(\theta - \sin\theta)^{2/3}}{4^{2/3}\,\theta^{2/3}} \cdot S_o^{1/2}$

$$\Rightarrow Q = \left(\frac{S_o^{1/2}}{n}\right)\left(\frac{D^{8/3}}{8(4)^{2/3}}\right)\left[\frac{(\theta - \sin\theta)^{5/3}}{\theta^{2/3}}\right]$$

* Here, $y = D/2\left[1 - \cos(\theta/2)\right] \Rightarrow \theta = 2\cos^{-1}\left(1 - \frac{2y}{D}\right)$

* Substitution gives,

$$Q = \left(\frac{S_o^{1/2}\,D^{8/3}}{n\,8\,(4)^{2/3}}\right) \frac{\left[2\cos^{-1}\left(1 - \frac{2y}{D}\right) - \sin\left\{2\cos^{-1}\left(1 - \frac{2y}{D}\right)\right\}\right]^{5/3}}{2^{2/3}\left[\cos^{-1}\left(1 - \frac{2y}{D}\right)\right]^{2/3}}$$



* In the shaded regime, two different flow depth produces similar flow rate.

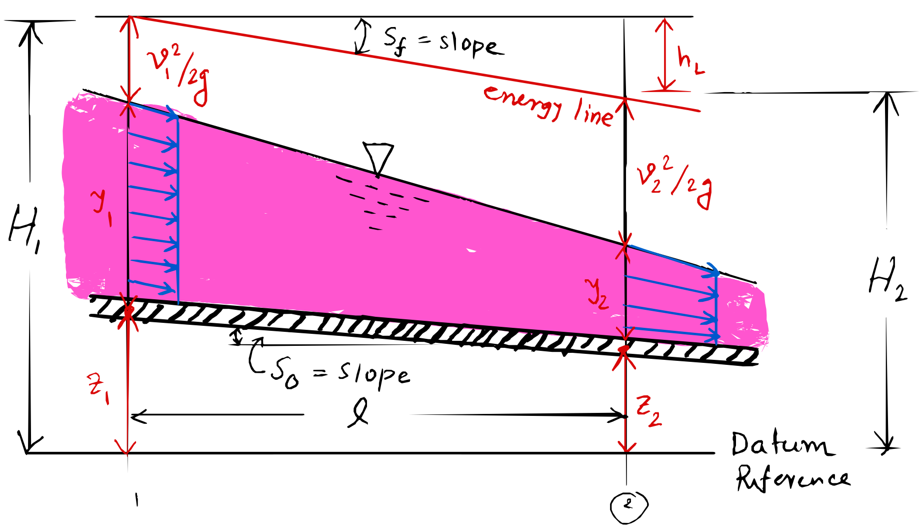
* with y increasing, A (flow area) increases but P (wetted perimeter) increases too.

$\begin{cases} A \rightarrow \text{increases flow rate} \\ P \rightarrow \text{increase shear stress (Decreases flow)} \end{cases}$

* The difference between Q_{full} and Q_{max} is less than 10%. Considering 'n' values have higher inaccuracy, it is often neglected.

Gradually varied Flow

* For Gradually varied flow $y = f(x)$ and $\frac{dy}{dx} \ll 1$.



* Energy conservation equation ritten for total head between section ① and ② is

$$H_1 = H_2 + h_L$$

Here, H represents total energy head at any section as $H = y + \frac{v^2}{2g} + z$ and h_L is frictional head loss between section ① and ②.

* From figure we see,

(a) $\dfrac{dH}{dx} = \dfrac{dh_L}{dx} = S_f$ (friction slope)

(b) $\dfrac{dz}{dx} = S_o$ (Bottom slope)

* $\left(\dfrac{dh_L}{dx}\right) = \dfrac{dH}{dx} = \dfrac{d}{dx}\left[y + \dfrac{v^2}{2g} + z\right]$ $\left\{\begin{matrix}\text{remember} \\ v = f(x)\end{matrix}\right\}$

$\Rightarrow S_f = \dfrac{dy}{dx} + \dfrac{v}{g}\dfrac{dv}{dx} + \dfrac{dz}{dx}$

$\Rightarrow \dfrac{v}{g}\dfrac{dv}{dx} + \dfrac{dy}{dx} = (S_f - S_o)$ ——— (I)

* Now we can obtain velocity from the flowrate as,

$v = \dfrac{Q}{A} = \left(\dfrac{q}{y}\right)$

$\Rightarrow \left(\dfrac{dv}{dx}\right) = q\left(-\dfrac{1}{y^2}\right)\dfrac{dy}{dx}$ ——— (II)

* Combining equation (I) and (II) we get

$\left(1 - \dfrac{qv}{y^2 g}\right)\dfrac{dy}{dx} = S_f - S_o$

$\Rightarrow \left[1 - \left(\dfrac{v^2}{yg}\right)\right]\dfrac{dy}{dx} = S_f - S_o$

* We recal that $F_r = \left(\frac{v}{\sqrt{gy}}\right)$ and substitute this definition into the above equation to get

$$\left(\frac{dy}{dx}\right) = \boxed{\left(\frac{S_f - S_o}{1 - F_r^2}\right)}$$

Geometry and flow condition

Chang in fluid flow depth

upstream Froude Number

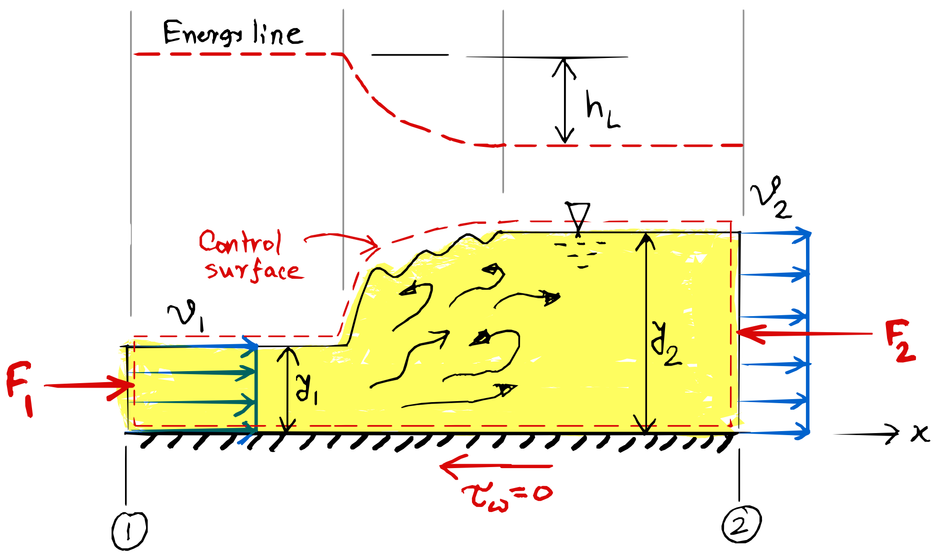
* what does above equation tells about change in flow depth?
  → Does the flow depth always increase $\left(\frac{dy}{dx} > 1\right)$ or always decrease $\left(\frac{dy}{dx} < 1\right)$

Interesting
* For $F_r < 1$ (Subcritical flow) the denominator $(1 - F_r^2)$ is alway positive.
* For $F_r > 1$ (Super critical flow) the denominator $(1 - F_r^2)$ is always negative.

* Think about $\frac{dy}{dx}$ being positive for $F_r < 1$. How is it possible to Increase flow depth?
  → Answer is "gravity", which causes the flow to decelerate and increase flow depth.

Rapidly varied Flow

* Flow is non uniform, but (dy/dx) is not small.
* often sudden change in flow depth is observed when fluid flows from shallow to deep channels.



* Momentum balance : $\int_{cs} \vec{v}\rho\vec{v}\cdot d\vec{A} = \sum \vec{F}$

$$\Rightarrow -\cancel{\rho} v_1^2 y_1 \cancel{b} + \cancel{\rho} v_2^2 y_2 \cancel{b} = \left(\cancel{\rho} g \frac{y_1}{2}\right) y_1 \cancel{b} - \left(\cancel{\rho} g \frac{y_2}{2}\right) y_2 \cancel{b}$$

$$\Rightarrow -v_1^2 y_1 + v_2^2 y_2^2 = \frac{1}{2} g y_1^2 - \frac{1}{2} g y_2^2$$

$$\Rightarrow \frac{y_1^2}{2} - \frac{y_2^2}{2} = \left(\frac{v_2^2 y_2}{g}\right) - \left(\frac{v_1^2 y_1}{g}\right)$$

$$\Rightarrow \frac{y_1^2}{2} - \frac{y_2^2}{2} = \left(\frac{v_1 y_1}{g}\right)(v_2 - v_1)$$

$$\boxed{\begin{array}{c}\text{mass balance} \\ v_1 y_1 b = v_2 y_2 b\end{array}}$$

* Energy balance: $(\tau_w = 0)$

Jump loss

$$y_1 + \frac{v_1^2}{2g} + \cancel{z_1} = y_2 + \frac{v_2^2}{2g} + \cancel{z_2} + h_L$$

$$\Rightarrow \quad y_1 + \left(\frac{v_1^2}{2g}\right) = y_2 + \left(\frac{v_2^2}{2g}\right) + h_L$$

* Listing momentum and energy balance equations:

$$\begin{cases} \text{Momentum: } \dfrac{y_1^2}{2} - \dfrac{y_2^2}{2} = \left(\dfrac{v_1 y_1}{g}\right)(v_2 - v_1) \\ \text{Energy: } y_1 - y_2 = \dfrac{1}{2g}(v_2^2 - v_1^2) + h_L \end{cases}$$

clearly there is a solution to above equations for $h_L = 0$, which is $y_1 = y_2$ and $v_1 = v_2$.

"This solution is called trivial solution and represent no-jump case!!

* Is there any other solution for $h_L \neq 0$?

Substitution of $v_2 = \left(\frac{y_1}{y_2}\right) v_1$ in momentum

$$\frac{y_1^2}{2} - \frac{y_2^2}{2} = \left(\frac{v_1 y_1}{g}\right)\left(\frac{v_1 y_1}{y_2} - v_1\right)$$

$$\Rightarrow \frac{1}{2}(y_1^2 - y_2^2) = \left(\frac{v_1^2 y_1}{g y_2}\right)(y_1 - y_2)$$

$$\Rightarrow \frac{1}{2}(y_1 + y_2) = \frac{v_1^2 y_1}{g y_2}$$ (Since we are looking for $y_1 \neq y_2$)

$$\Rightarrow \frac{y_1}{2} + \frac{y_2}{2} - \frac{v_1^2 y_1}{g y_2} = 0$$

$$\Rightarrow \frac{y_1 y_2 g + y_2^2 g - 2 v_1^2 y_1}{2 y_2 g} = 0$$

$$\Rightarrow y_2^2 g + y_2 y_1 g - 2 v_1^2 y_1 = 0$$

$$\Rightarrow \left(\frac{y_2}{y_1}\right)^2 + \left(\frac{y_2}{y_1}\right) - \left(\frac{2 v_1^2}{y_1 g}\right) = 0$$

$$\Rightarrow (y_2/y_1)^2 + (y_2/y_1) - 2 F_{r_1}^2 = 0$$

* Solution to above quadratic equation is

$$\left(\frac{y_2}{y_1}\right) = \frac{-1 \pm \sqrt{1 + 8 F_{r_1}^2}}{2}$$ (F_{r_1} = upstream Froude number)

Since, $\sqrt{1+8F_r^2} > 1$, the negative sign would produce $y_2/y_1 < 0$ (unphysical).

Thus,

$$\left(\frac{y_2}{y_1}\right) = \frac{1}{2}\left(-1 + \sqrt{1+8F_{r_1}^2}\right)$$

* (for $F_r > 0$, y_1 and y_2 ratio) *

* what happens when $0 < F_r < 1$?

→ The above expression has no problem (momentum equation is ok)

* No from energy equation, $v_2 = \left(\frac{v_1 y_1}{y_2}\right)$ gives

$$y_1 - y_2 = \frac{1}{2g}\left(v_2^2 - v_1^2\right) + h_L$$

$$\Rightarrow y_1 - y_2 = \frac{1}{2g}\left(\frac{v_1^2 y_1^2}{y_2^2} - v_1^2\right) + h_L$$

$$\Rightarrow h_L = (y_1 - y_2) - \left(\frac{v_1^2 y_1^2}{2g y_2^2}\right) + \left(\frac{v_1^2}{2g}\right)$$

$$\Rightarrow \left(\frac{h_L}{y_1}\right) = 1 - \left(\frac{y_2}{y_1}\right) + \frac{v_1^2}{2g y_1} - \frac{v_1^2 y_1}{2g y_2^2}$$

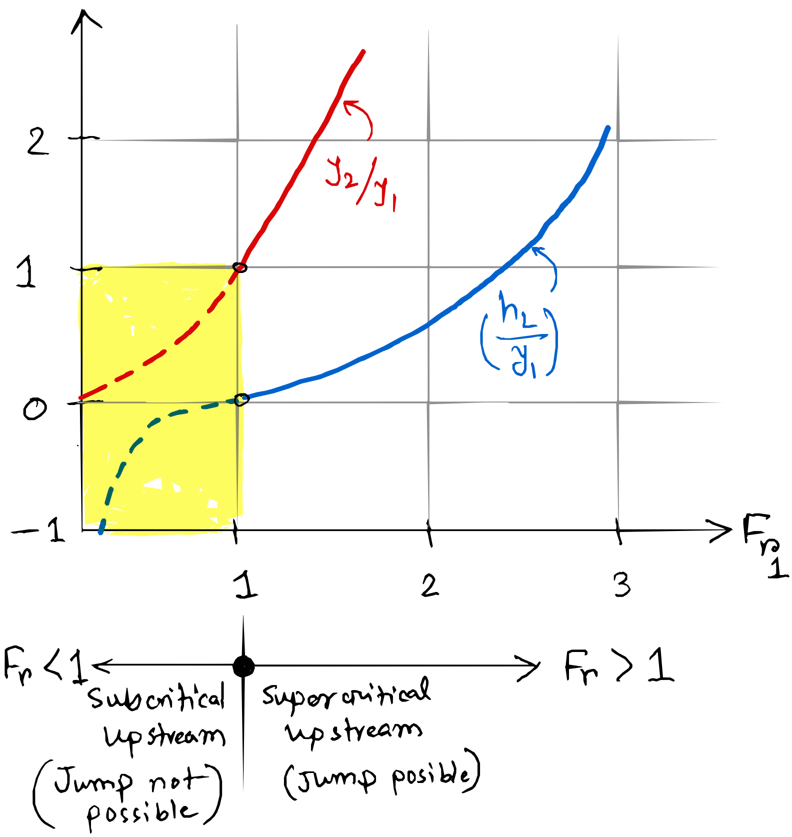
$$\Rightarrow \left(\frac{h_L}{y_1}\right) = 1 - \left(\frac{y_2}{y_1}\right) + \frac{v_1^2}{2g y_1}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right]$$

$$\Rightarrow \left(\frac{h_L}{y_1}\right) = 1 - \left(\frac{y_2}{y_1}\right) + \left(F_{r_1}^2/2\right)\left[1 - \left(y_1/y_2\right)^2\right]$$

* Replacing $\left(y_2/y_1\right) = \frac{1}{2}\left(-1 + \sqrt{1 + 8F_{r_1}^2}\right)$ we get

$$\left(\frac{h_L}{y_1}\right) = 1 - \frac{1}{2}\left(-1 + \sqrt{1 + 8F_{r_1}^2}\right) + \frac{F_{r_1}^2}{2}\left[1 - \frac{4}{\left(-1 + \sqrt{1 + 8F_{r_1}^2}\right)^2}\right]$$

How does this expression look ?

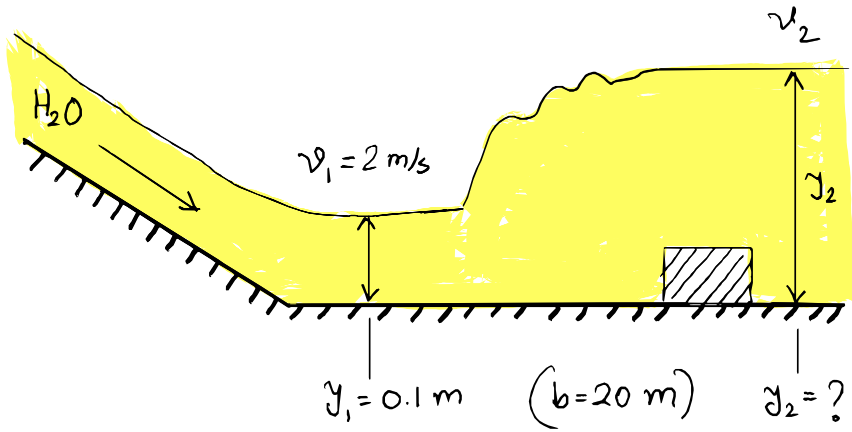


* The figure reveals that for $F_{r_1} < 1$ the $h_L < 0$ (which is unphysical).

* Jump is only possible if the upstream flow is supercritical (inertia dominated flow).

* During jump energy is dessipated from the flow as turbulance. (Useful)

*** Example:** $h_L = ?$ Power loss $= ?$, $Fr_2 = ?$



* First we check upstream flow Froude number.

$$Fr_1 = \left(\frac{v_1}{\sqrt{g y_1}}\right) = \frac{2}{\sqrt{9.8 \times 0.1}} = 2.02 \quad \text{(super critical)}$$

→ Jump possible.

* $(y_2/y_1) = \frac{1}{2}\left(-1 + \sqrt{1 + 8 Fr_1^2}\right) = 2.4$, $y_2 = 0.24\ m$ (Ans)

* Mass balance, $v_2 = \left(\frac{v_1 y_1}{y_2}\right) = 0.83\ m/s$

* $$(h_L/y_1) = 1 - \left(\frac{y_2}{y_1}\right) + \frac{Fr_1^2}{2}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right]$$

$$= 1 - 2.4 + \frac{2.02^2}{2}\left[1 - (1/2.4)^2\right] = 0.286$$

$\Rightarrow h_L = 0.0286\ m.$ (Ans)

* Flow rate, $Q = v_1 b y_1 = 2 \times 20 \times 0.1 \ m^3/s$

$$\Rightarrow Q = 4.0 \ m^3/s$$

* Power loss, $P_{loss} = \rho g Q h_L = (9800 \times 4 \times 0.0286) \ W$

$$\Rightarrow P_{loss} = 1121 \ W \approx 1.12 \ kW$$

(Ans)

* Froude number at down stream

$$Fr_2 = \frac{v_2}{\sqrt{g y_2}} = \frac{0.83}{\sqrt{9.8 \times 0.24}} = 0.54 \ (\text{subcritical})$$