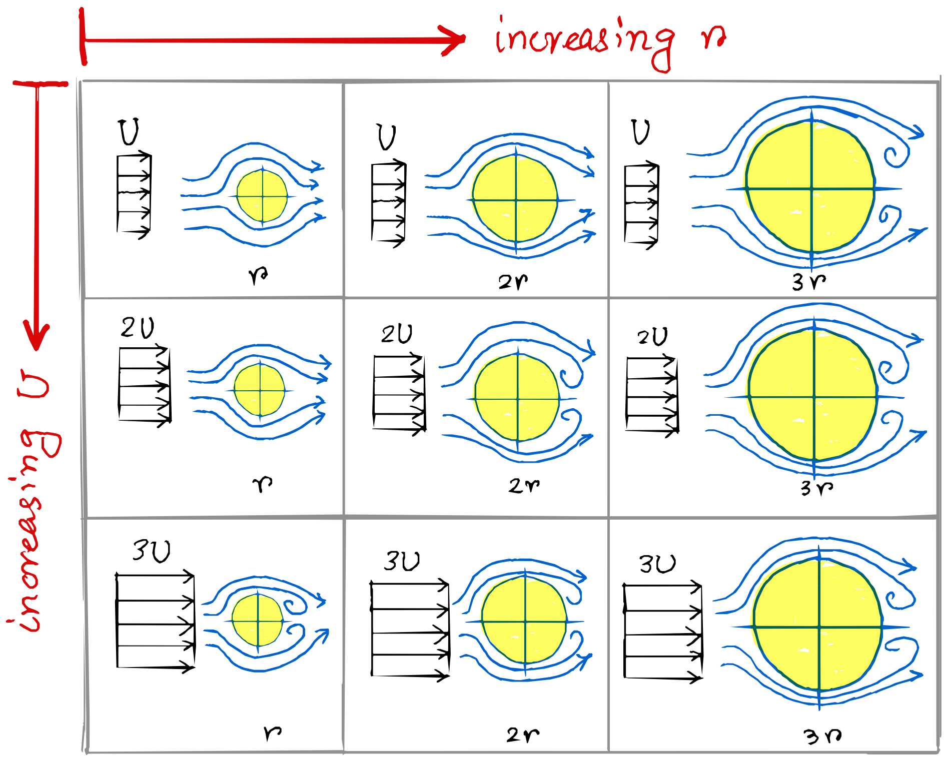
Ch-7

* Dimensional Analysis

→ Collection of parameters to form non-dimensional numbers (Groups)

* Think about flow of different fluid over solid cilynders of different size at diffent velocity.



* Think changing the fluid ($\nu = \mu/\rho$)

* There are 3-parameters that affect the flow charecteristics

(a) free stream velocity, U (m/s)
(b) Cylinder diameter, d (m)
(c) fluid's kinematic viscosity, ν (m^2/s)

* How can one combine these 3 parameters to create a non dimensional (unitless) parameter.

$$\left(\frac{Ud}{\nu}\right) \longrightarrow \frac{(m/s)\,m}{m^2/s} \longrightarrow \left(\frac{m^2/s}{m^2/s}\right) \quad ✓$$

$$(U d \nu) \longrightarrow (m/s)\,m\,(m^2/s) \longrightarrow (m^4/s^2) \quad \times$$

* These mean $\left(Ud/\nu\right)$ or $\left(\frac{\rho U d}{\mu}\right)$ is one group of parameters that is necessary to describe the flow.

$\longrightarrow$ How about $\left(\frac{\nu}{Ud}\right)$?

$\longrightarrow$ How about $\left(\frac{\nu^2}{U^2 d^2}\right)$?

* Big question: How many such non dimensional number we need to explain (fully describe) the flow characteristics?

Backingham Pi - theorem

"If an equation involving "K" no of variables is dimensionally homogenous, it can be reduced to a relationship among "K-p" independent dimensionless products, where "p" is the minimum number of reference dimensions required to describe the variables"

* The "independent dimensionless produts" are called Pi-Terms.

* How to obtain Pi-terms?
  - → Method of repeating variables (9-steps)

* Step-1: list all the variables involved
  * Geometry (diameter, lengths etc)
  * Properties (Density, viscosity etc)
  * External Effects (velocity, pressure etc)
  * Time (time, rates)

(All variables need to be independent)

diameter $\longleftrightarrow$ area

density $\longleftrightarrow$ sp. gravity

Step-2: Select a system of dimensions to use
* (MLT or FLT) ⟶ Do not interchange

Step-3: Determine required number of π-terms
* How many non-dimensional groups are required to fully describe the system.

Step-4: Select the repeating variables
* Dimensionally independent
* Independent variables only

Step-5: Form π-term(s)
* Group each non-repeating variable(s) with the repeating variable(s) raised to arbitrary exponent.

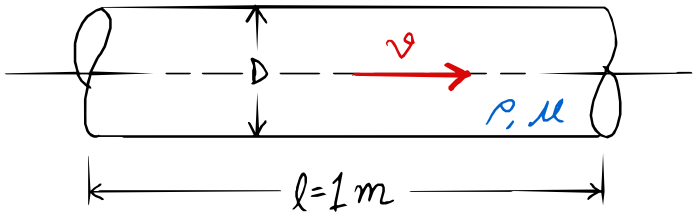
Step-6: Determine the arbitrary exponents from dimensional homogenity.

Step-7: Check back that all π-terms are dimensionless

Step-8: Try to use standard Non-Dimensional numbers (if possible)

Step-9: Express the functional form.

* Example (Flow through tube)



Step-1: list all variable(s) that are independent

(a) Geometry $\longrightarrow$ D $\qquad$ A, P_m, $\forall$

(b) Properties $\longrightarrow$ ρ, μ $\qquad$ ν, SG, m

(c) External $\longrightarrow$ ϑ, $\Delta P/l$ $\qquad$ $\dot{q}$, $\dot{m}$

(d) Time $\longrightarrow$ None $\qquad$ t (l/ϑ)

Six variables $\longrightarrow$ $(K=5)$

Step-2: Dimensions of parameters (MLT)

$$\left.\begin{aligned} D &\longrightarrow L \\ \rho &\longrightarrow ML^{-3} \\ \mu &\longrightarrow ML^{-1}T^{-1} \\ \vartheta &\longrightarrow LT^{-1} \\ \left(\frac{\Delta P}{l}\right) &\longrightarrow ML^{-2}T^{-2} \end{aligned}\right\}$$

3-dimensions
(M, L T)
$(r = 3)$

Step-3: Pi-Terms needed $(5-3) = 2$

Step-4: selected repeated variables are

* D (l is linearly dependent)
* ρ (Not linearly dependent)
* v (ΔP is dependent variable)

Step-5 and 6:

First π-term, $\pi_1 = \left(\frac{\Delta P}{L}\right) D^a \rho^b v^c$

$$\Rightarrow (MLT)^0 = (ML^{-2}T^{-2})(L)^a(ML^{-3})^b(LT^{-1})^c$$

$$\Rightarrow M^0L^0T^0 = M^{1+b}\,L^{-2+a-3b+c}\,T^{-2-c}$$

Thus, $0 = 1+b,\ b=-1$ (M)

$0 = -2-c,\ c = -2$ (T)

$0 = -2+a-3b+c,\ a = 1$ (L)

$$\pi_1 = \left(\frac{\Delta P}{L}\right)\left(\frac{D}{\rho v^2}\right)$$

Second π-term, $\pi_2 = \mu D^a \rho^b v^c$

$$\Rightarrow M^0L^0T^0 = (ML^{-1}T^{-1})(L)^a(ML^{-3})^b(LT^{-1})^c$$

Thus, $0 = 1+b,\ b=-1$ (M)

$0 = -1-c,\ c=-1$ (T)

$0 = -1+a-3b+c,\ a=-1$ (L)

$$\pi_2 = \mu\cdot\left(\frac{1}{\rho v D}\right) = \left(\frac{\mu}{\rho v D}\right)$$

Step-7: $\pi_1 = \left(\frac{\Delta P}{L}\right)\left(\frac{D}{\rho v^2}\right) = (ML^{-2}T^{-2})(L)(ML^{-3})^{-1}(LT^{-1})^{-2}$

$= M^0L^0T^0$ (checked)

$\pi_2 = \left(\frac{\mu}{\rho v D}\right) = (ML^{-1}T^{-1})(ML^{-3})^{-1}(LT^{-1})^{-1}(L)^{-1}$

$= M^0L^0T^0$ (checked)

Step-8: π_2 term is recognizable as Raynolds number (inverse)

$$\pi_2 = \left(\frac{\rho v D}{\mu}\right) = Re$$

Step-9: The functional form reveals that

$$\pi_1 = f(\pi_2)$$

$$\Rightarrow \left(\frac{\Delta P}{L}\right)\left(\frac{D}{\rho v^2}\right) = f\,(Re)$$

$$\Rightarrow \left(\frac{\Delta P}{L}\right)\left(\frac{D}{\rho v^2}\right) = f\left(\frac{\rho v D}{\mu}\right)$$

* Dimensional analysis is not capable of determing the exact functional form.

* The above relation shows that

$$\underbrace{(\Delta P)}_{\text{(Net pressure drop)}}\underbrace{\left(\frac{D}{L}\right)}_{\text{(Tube aspect ratio)}}\Big/\underbrace{(\rho v^2)}_{\text{(2 × Dynamic pressure)}} = f\,\underbrace{(Re)}_{\text{(Reynolds number)}}$$

* later we will see $(\Delta P/\rho v^2)$ is called Euler number

$$\underbrace{(Eu)\left(\frac{D}{L}\right)}_{\text{Scaled Euler number}} = f\,(Re) \leftarrow \text{Reynolds number}$$

Some commonly used in fluid mechanics

(a) Reynolds number, $Re = \left(\frac{\rho V D}{\mu}\right)$

$$Re = \left(\frac{\text{Inertia force}}{\text{Viscous force}}\right)$$

(b) Froude number, $F_r = \left(\frac{V}{\sqrt{gl}}\right)$

$$F_r = \left(\frac{\text{Inertia force}}{\text{Graviti force}}\right)$$

(c) Euler number, $Eu = \left(\frac{P}{\rho V^2}\right)$

$$Eu = \left(\frac{\text{Pressure force}}{\text{Inertia force}}\right)$$

(d) Mach number, $Ma = \left(\frac{V}{c}\right)$

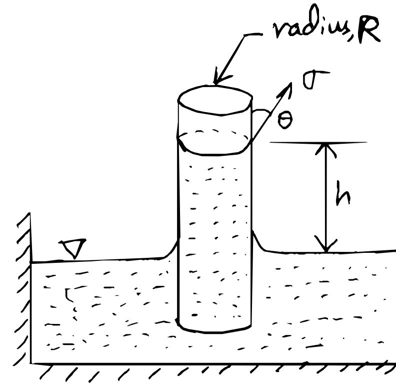
$$Ma = \left(\frac{\text{Inertia force}}{\text{compressibility force}}\right)$$

(e) Weber number, $We = \left(\frac{\rho V^2 l}{\sigma}\right)$

$$We = \left(\frac{\text{Inertia force}}{\text{Surface tension force}}\right)$$

"See table 7.1 (Page-362) of textbook"

Capilary rise in tube



* The important variables are
  $R, \theta, \sigma, \gamma = \rho g$

  (ρ and g are not expected to influence the h individually)

* $h = f(R, \theta, \sigma, \gamma)$

  $K = 5, \quad r = 2 \quad (M, L)$

  3 π-terms

* Repeating variable (R, σ)

  $\pi_1 = h R^a \sigma^b \longrightarrow \pi_1 = \left(h/R\right)$

  $\pi_2 = \theta R^a \sigma^b \longrightarrow \pi_2 = (\theta)$

  $\pi_3 = \gamma R^a \sigma^b \longrightarrow \pi_3 = \left(\frac{\gamma R^2}{\sigma}\right)$

* $\frac{h}{R} = f\left(\theta, \frac{\gamma R^2}{\sigma}\right)$

* actual solution, $h = \left(\frac{2\sigma \cos\theta}{\gamma R}\right)$

$$\Rightarrow \left(\frac{h}{R}\right) = 2\,(\cos\theta)\left\{\frac{1}{(\gamma R^2/\sigma)}\right\}$$

$$\Rightarrow \left(\frac{h}{R}\right) = \frac{1}{Bo}\, 2\cos\theta$$

Here $Bo = \left(\frac{\gamma R^2}{\sigma}\right) = \left(\frac{\rho g R}{\sigma / R}\right) = \left(\frac{\text{Gravity}}{\text{Surface tension}}\right)$

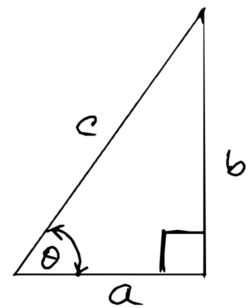
* Pythagoras from non-dimensional analysis

(Just for FUN)

* Area, $A = f(a, b)$

  or, $A = f(a, c)$

  or, $A = f(c, \theta)$



* We use $A = f(c, \theta)$

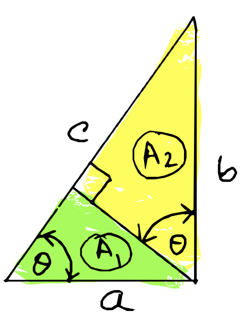
  $K = 3$ and $r = 1$ (L)

  "2 - π terms"

* $\left.\begin{array}{l} \pi_1 = A/c^2 \\ \pi_2 = \theta \end{array}\right\}$ use 'c' as repeating variable

* $\dfrac{A}{c^2} = f(\theta)$

* Now we do the samething by dividing the triangle into 2-right angle triangle as below.



$\dfrac{A_1}{b^2} = f(\theta)$, $\quad \dfrac{A_2}{a^2} = f(\theta)$

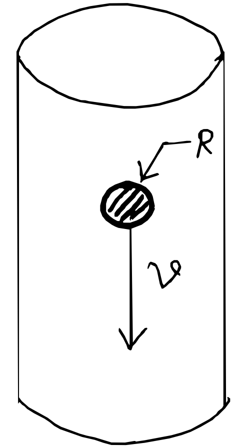
Now, $A = A_1 + A_2$

$\Rightarrow c^2 f(\theta) = b^2 f(\theta) + a^2 f(\theta)$

$\Rightarrow c^2 = a^2 + b^2$

* Free falling sphere in liquid

* Force depends on v, R, μ
  (Not on density !!)



* $F = f(v, R, \mu)$

  $K = 4$, $r = 3$ (MLT)

  π-term $= 1$

* Repeating variable R, v, μ

$$\pi_1 = F \cdot R^a v^b \mu^c$$

$$(MLT^{-2}) \quad (L)^a \quad (LT^{-1})^b \quad (ML^{-1}T^{-1})^c$$

$$\left.\begin{array}{ll} 1+c=0, \; c=-1 & (M) \\ 1+a+b-c=0 & (L) \\ -2-b-c=0 & (T) \end{array}\right\} \quad \begin{array}{l} a=-1 \\ b=-1 \\ c=-1 \end{array}$$

$$\pi_1 = \left(\frac{F}{\mu R v}\right)$$

* $\pi_1 =$ function of nothing (constant)

$\Rightarrow F = (\text{const}) \, \mu R v$

* From actual solution, $F = 6\pi \mu R v$.
  $\uparrow$ (constant)

Prototype	Model
* Has exact geometric size & imposed cond. (exact replica)	* Scalled up/down version of exact replica. (toy version)
* Rarely used for testing (Expensive test)	* Often used for testing. (Tunnel test)

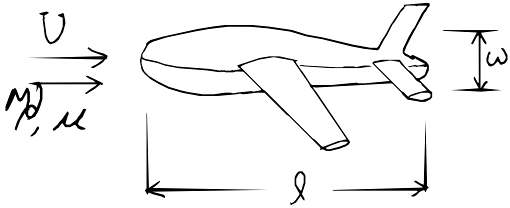
* Similitude ⟶ scalling analysis allow experiment to be done on models and interprete the data for real applications.

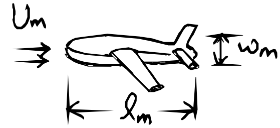
** tool: Non dimensional number

↓

This is simple, works like magic !!
(May be confusing for begineers)

* External flow over solid body:





model

* Drag force (is manifestation of pressure). D

variables, U, l, w, D, ρ, μ $(K=6)$

$$\begin{array}{cccccc} U & l & w & D & \rho & \mu \\ \downarrow & \downarrow & \downarrow & \downarrow & \searrow & \searrow \\ L/T & L & L & MLT^{-2} & ML^{-3} & ML^{-1}T^{-1} \end{array}$$

$(r=3)$

* Need 3 Pi-terms

* Repeating variables l, U, ρ

* $\pi_1 = w \cdot l^a U^b \rho^c \longrightarrow M^0L^0T^0 = (L)(L)^a(LT^{-1})^b(ML^{-3})^c$

$\Rightarrow 0 = c$ (M)

$\Rightarrow 0 = -b$ (T)

$\Rightarrow 0 = 1 + a - b - 3c, \quad a = -1$

$\pi_1 = (w/l)$

* $\pi_2 = D \cdot l^a U^b \rho^c \longrightarrow M^0L^0T^0 = (MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$

$\Rightarrow 0 = 1 + c, \quad c = -1$ (M)

$\Rightarrow 0 = -2 - b, \quad b = -2$ (T)

$\Rightarrow 0 = 1 + a + b - 3c, \quad a = -2$

$\pi_2 = \left(\dfrac{D}{\rho U^2 l^2}\right)$

* $\pi_3 = \mu\, l^a U^b \rho^c \longrightarrow M^0L^0T^0 = (ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c$

$\Rightarrow 0 = 1 + c, \quad c = -1$ (M)

$\Rightarrow 0 = -1 - b, \quad b = -1$ (T)

$\Rightarrow 0 = -1 + a + b - 3c, \quad a = -1$

$\pi_3 = \left(\dfrac{\mu}{\rho U l}\right)$

Thus, $\pi_2 = f(\pi_1, \pi_3)$

$$\Rightarrow \underbrace{\left(\frac{D}{\rho U^2 l^2}\right)}_{\text{Drag coefficient}} = f\Big(\underbrace{\frac{w}{l}}_{\text{aspect ratio}}, \underbrace{\frac{\mu}{\rho U l}}_{\text{Reynolds number}}\Big)$$

* For model and prototype:

$$\left(\frac{D}{\rho U^2 l^2}\right) = \left(\frac{D_m}{\rho_m U_m^2 l_m^2}\right) \xrightarrow[\text{Interest}]{} (1^{st} \text{ similitude})$$

$$\left(\frac{w}{l}\right) = \left(\frac{w_m}{l_m}\right) \xrightarrow[\text{Geometry}]{} (2^{nd} \text{ similitude})$$

$$\left(\frac{\mu}{\rho U l}\right) = \left(\frac{\mu_m}{\rho_m U_m l_m}\right) \xrightarrow[\text{External effect}]{} (3^{rd} \text{ similitude})$$

Model to prototype

* If the model is scalled down to 1/100 size and tested in water that produces 10 N of drag force, then what will be the drag force in the prototype flying in the air @ same speed of the model.