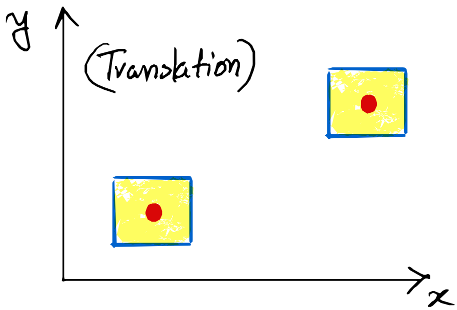
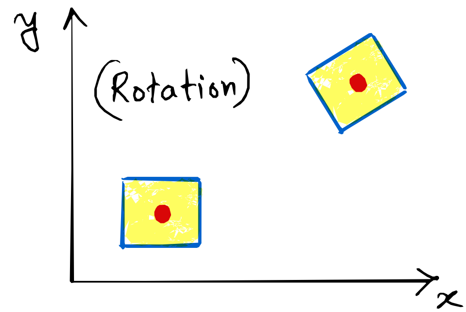
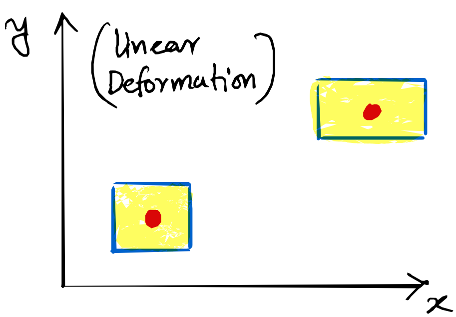
Ch-6

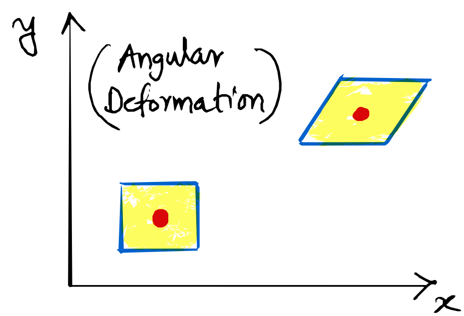
Motion & Deformation of "fluid element"

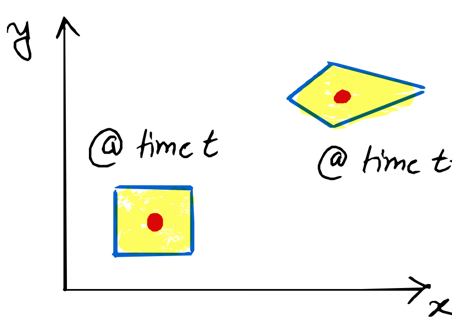
Fluid element: chunk of fluid that is hypothetical
continuous and homogeneous with
continuum properties.



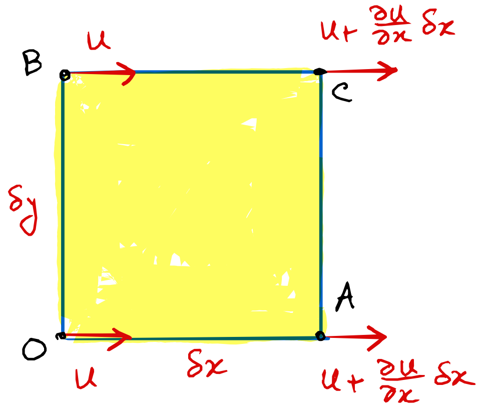


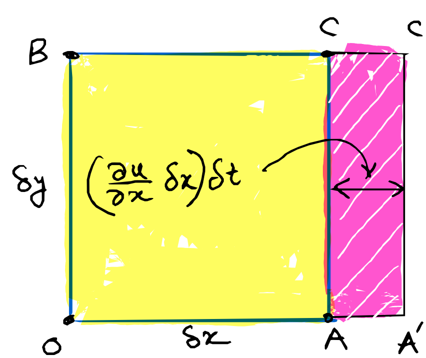






linear motion & deformation





Volume change $\delta \forall = \left[\left(u + \frac{\partial u}{\partial x}\delta x\right) - u\right]\delta t\, \delta y\, \delta z$

$$= \left(\frac{\partial u}{\partial x}\right)\delta t\, \delta x\, \delta y\, \delta z$$

rate of change of fractional volume,

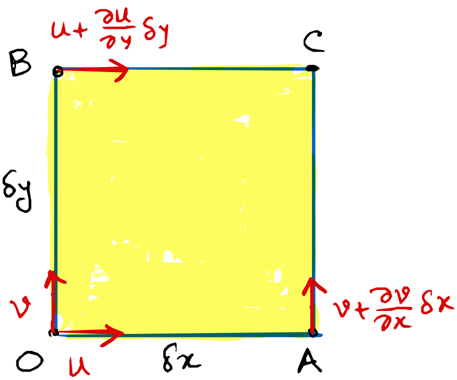
$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \lim_{\delta t \to 0}\left\{\frac{\left(\frac{\partial u}{\partial x}\right)\delta t}{\delta t}\right\} = \left(\frac{\partial u}{\partial x}\right)$$

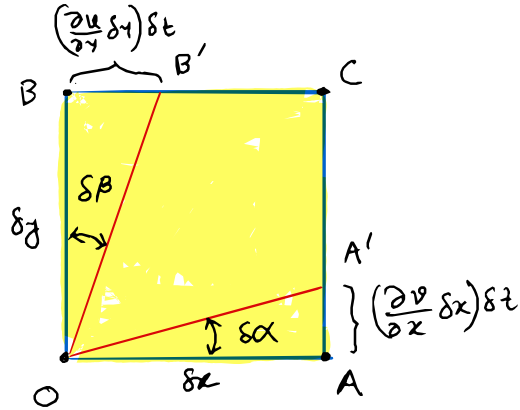
Similarly it can be shown, if velocity gradient is present in other two directions then,

$$\underline{\frac{1}{\delta \forall}\left(\frac{d\,\delta \forall}{dt}\right)} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \underline{\nabla \cdot \vec{v}}$$

(Divergence of $\vec{v}$ is Volumetric dilation rate)

Angular motion & deformation





Angular velocity of line OA,

$$\omega_{OA} = \lim_{\delta t\to 0}\left(\frac{\delta\alpha}{\delta t}\right) = \lim_{\delta t\to 0}\left(\frac{\tan\delta\alpha}{\delta t}\right) = \lim_{\delta t\to 0}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial v}{\partial x}$$

Similarly, $\omega_{OB} = \lim_{\delta t\to 0}\left(\frac{\delta\beta}{\delta t}\right) = \lim_{\delta t\to 0}\left(\frac{\tan\delta\beta}{\delta t}\right) = \lim_{\delta t\to 0}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial y}$.

Now how to obtain the angular speed of the fluid element?

$$\omega_z = (\omega_{OA} - \omega_{OB})/2 = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

* for other two planes it can be shown that

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

* we can construct a vector as

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \frac{1}{2}\nabla\times\vec{V}$$

* The angular speed of the fluid element rotation is half the curl of the velocity field.

* The curl of velocity field is called vorticity.

$$\vec{\xi} = 2\vec{\omega} = \nabla \times \vec{v}$$

Things to remmember:

① For incompressible flow, $\nabla \cdot \vec{v} = 0$

(No volumetric dialation of fluid element)

② For irrotational flow, $\nabla \times \vec{v} = 0$

(No vorticity in flow)

(No angular deformation of fluid element)

* Check the velocity field below

$$\vec{v} = (x^2 - y^2)\hat{i} - 2xy\,\hat{j}$$

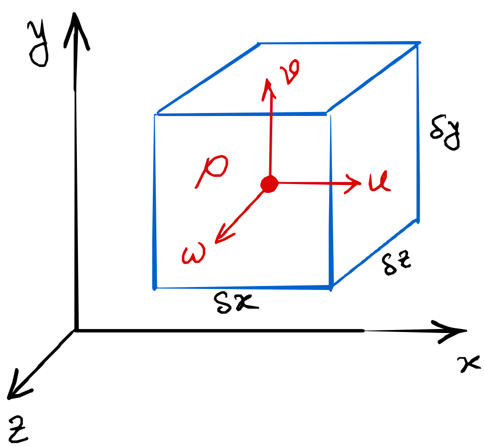
→ Is it an compressible/incompressible flow?

→ Is it a rotational/irrotational flow?

Conservation of mass

* Integral form: $\int_{cv} \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} \, d\forall = 0$

* Question: can we write $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$?

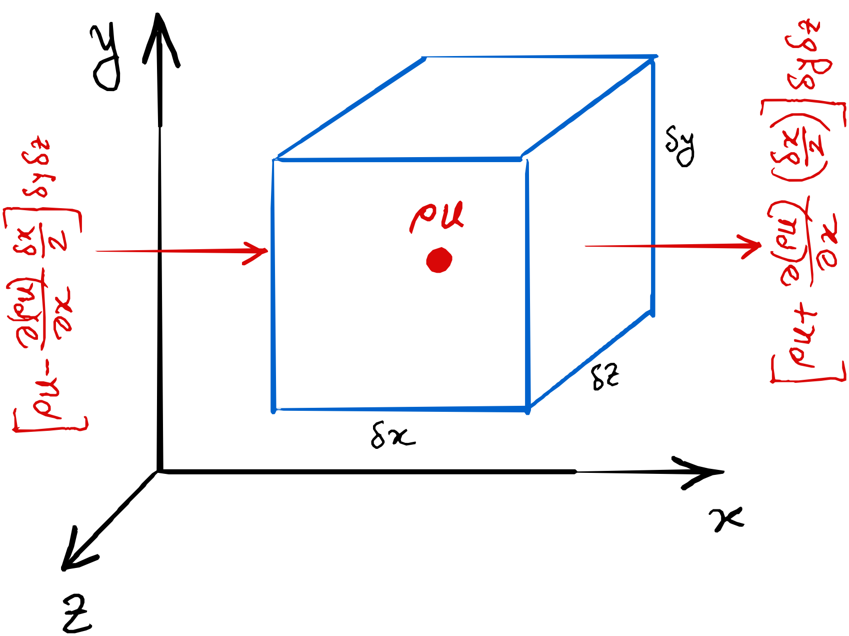


* How much mass is moved per unit area?

Density ρ, velocity u, $\Big\}$ ρu ??

$$\downarrow$$

$$\frac{kg}{m^3} \cdot \frac{m}{s} \rightarrow \left(\frac{kg}{m^2 \cdot s}\right)$$



* $\begin{bmatrix}\text{rate of change}\\ \text{in mass}\end{bmatrix} + \begin{bmatrix}\text{rate of mass}\\ \text{outflow}\end{bmatrix} = 0$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\,\delta\forall) + \left(\frac{\partial \rho u}{\partial x}\right)\delta x\,\delta y\,\delta z + \left(\frac{\partial \rho v}{\partial y}\right)\delta x\,\delta y\,\delta z + \left(\frac{\partial \rho w}{\partial z}\right)\delta x\,\delta y\,\delta z = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$ (This is true)

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

(Unsteady, compressible mass balance)

* Steady compressible flow, $\nabla\cdot(\rho\vec{v}) = 0$

* Steady incompressible flow, $\nabla\cdot\vec{v} = 0$

Stream function (ψ)

* steady, incompressible, 2-D flow:

$$\nabla \cdot \vec{V} = 0$$

$$* \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{———————} \quad (1)$$

* Eqn (1) relates u and v in such flows

* One trick is to assume

$$u = \frac{\partial \psi}{\partial y} \quad \& \quad \vec{v} = -\frac{\partial \psi}{\partial x}$$

* It automatically satisfy $\nabla \cdot \vec{V} = 0$

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
&\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0 \\
&\Rightarrow \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \\
&\Rightarrow 0 = 0
\end{aligned}\right\} \text{check on } \nabla \cdot \vec{V} = 0$$

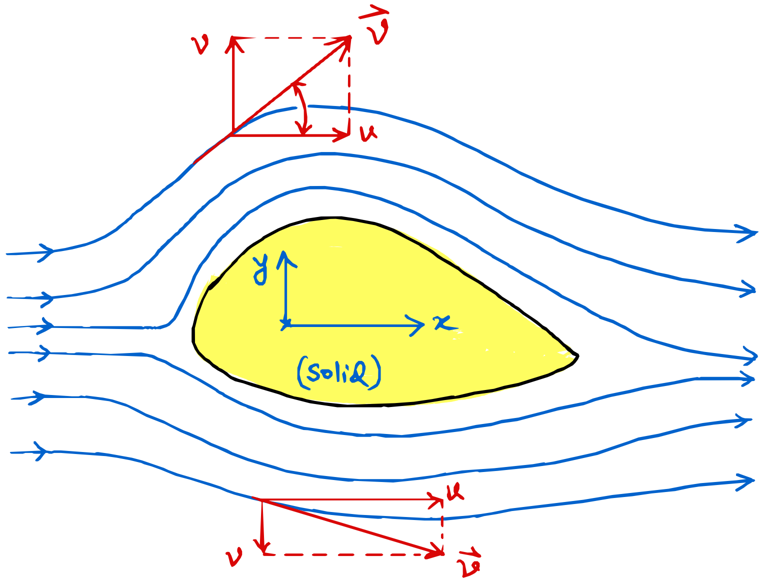
* what ψ gives?

$$d\psi = \left(\frac{\partial \psi}{\partial x}\right) dx + \left(\frac{\partial \psi}{\partial y}\right) dy = -v\,dx + u\,dy$$

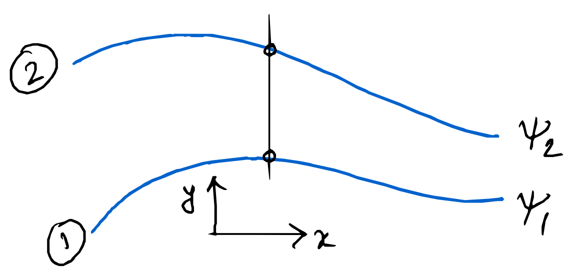
For a constant ψ line, $d\psi = 0$

$\rightarrow$ Thus, $\dfrac{dy}{dx} = \left(\dfrac{v}{u}\right)$

constant ψ lines are streamlines !!



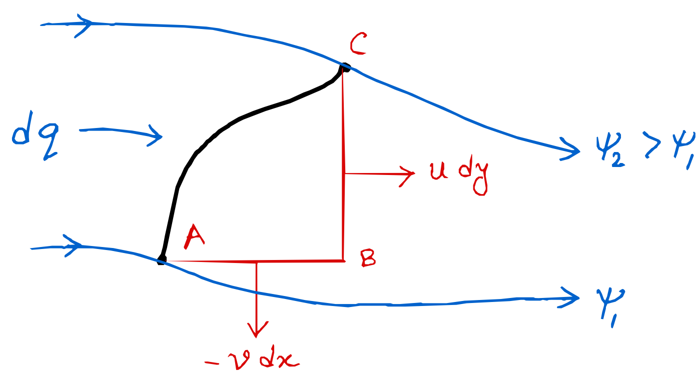
* Constant stream function lines are streamlines
* Significance of stream function values



* if $\psi_1 > \psi_2$ which way flow occures?

$$u = \frac{\partial \psi}{\partial y} = \left(\frac{\psi_2 - \psi_1}{y_2 - y_1}\right) < 0 \quad (u < 0 \; \leftarrow)$$

* if $\psi_2 > \psi_1$, $u = \frac{\partial \psi}{\partial y} = \left(\frac{\psi_2 - \psi_1}{y_2 - y_1}\right) > 0 \quad (u > 0 \; \rightarrow)$



$$dq = u\,dy - v\,dx$$

$$\Rightarrow dq = \left(\frac{\partial\psi}{\partial y}\right)dy + \left(\frac{\partial\psi}{\partial x}\right)dx = d\psi$$

* $q = \int_{\psi_1}^{\psi_2} d\psi = (\psi_2 - \psi_1) \longrightarrow$ Flow rate

* Actual value of ψ for a single streamline has no physical significance.

* Spacing between streamlines & the change of ψ values on streamlines have meaning.

Polar coordinate & continuity eqⁿ

$$\nabla\cdot\vec{v} = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0$$

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \& \quad v_\theta = -\frac{\partial \psi}{\partial r}}$$

$$* \; \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} = 0$$

$$\Rightarrow 0 = 0$$

check

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* How to obtain ψ from $\vec{v}$?

Given, $\vec{v} = 2y\,\hat{i} + 4x\,\hat{j} \longrightarrow u = 2y,\ v = 4x$

$$u = \frac{\partial \psi}{\partial y} = 2y \Rightarrow \psi = y^2 + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = 4x \Rightarrow \psi = -2x^2 + f(y)$$

Superposition of solution $\Big\}$ $\psi = y^2 - 2x^2 + C$

C ← arbitrary constant

Assume, $C = 0$, $\psi = y^2 - 2x^2$

(Example 6.3)

Velocity potential

* We know, for an irrotational flow $\nabla \times \vec{v} = 0$

"If a vector field is curl free ($\nabla \times \vec{v} = 0$) then there exist a scalar function ϕ (called potential function) whose gradient gives the vector field ($\vec{v} = \nabla \phi$)"

* if $\nabla \times \vec{v} = 0$ then $\vec{v} = \nabla \phi$

Thus, $\vec{v} = u\hat{i} + v\hat{j} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}$

$\therefore\ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$

* stream function ψ: $u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$

$\left.\frac{dy}{dx}\right|_{\psi = const} = \left(\frac{v}{u}\right)$ ← (streamline)

* $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u\,dx + v\,dy$

$\Rightarrow \left.\frac{dy}{dx}\right|_{\phi = const} = -\left(\frac{u}{v}\right)$ ← (equipotential line)

Thus, $\left(\frac{dy}{dx}\right)_{\psi = const} \cdot \left(\frac{d\phi}{dx}\right)_{\phi = const} = -1$

* For velocity field, $\vec{v} = 1\hat{i} - 1\hat{j}$ determine ψ & ϕ

Stream function ψ: $u = \frac{\partial \psi}{\partial y} = 1$, $\psi = y + f(x)$

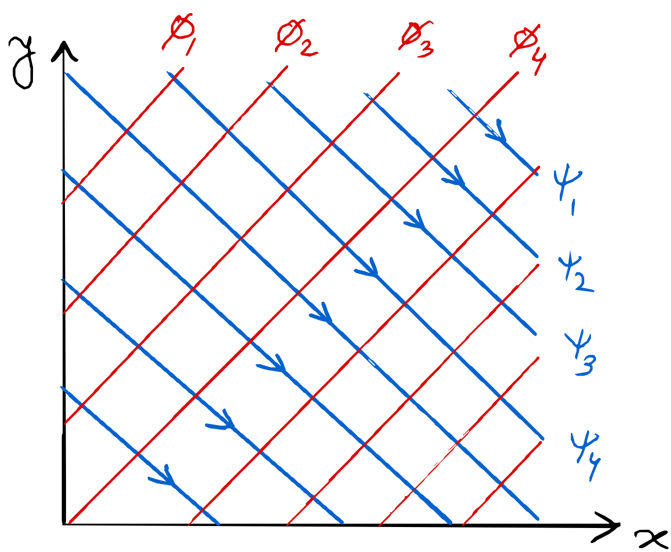
$v = -\frac{\partial \psi}{\partial x} = -1$, $\psi = x + f(y)$

Thus, $\psi = x + y + c \longrightarrow \psi = x + y$

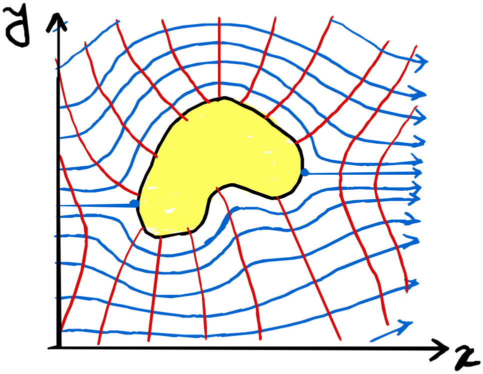
velocity potential ϕ: $u = \frac{\partial \phi}{\partial x} = 1$, $\phi = x + f(x)$

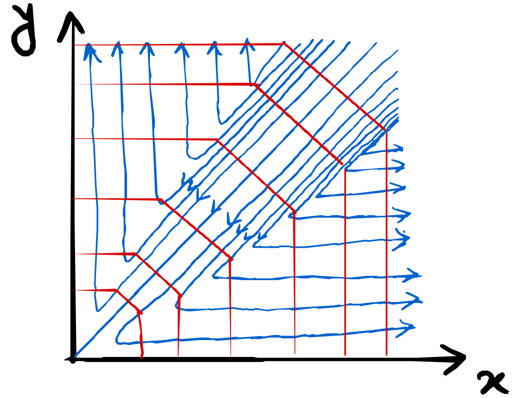
$v = \frac{\partial \phi}{\partial y} = -1$, $\phi = -y + f(y)$

Thus, $\phi = x - y + c \longrightarrow \phi = x - y$



* ϕ and ψ are always orthogonal (perpendicular)





Cartesian to polar co-ordinate

* Cartesian : $\vec{v} = u\,\hat{i} + v\,\hat{j}$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$\Updownarrow$$

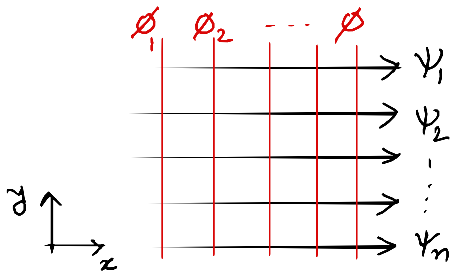
* Polar : $\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Some basic potential flow

① uniform flow:



$$\vec{v} = U\hat{i} + 0\hat{j}$$

$$\frac{\partial \psi}{\partial y} = U \quad , \quad -\frac{\partial \psi}{\partial x} = 0$$

$$\psi = Uy + f(x) \qquad \psi = f(y)$$

$$\psi = Uy + \cancel{C}^{\,0}$$

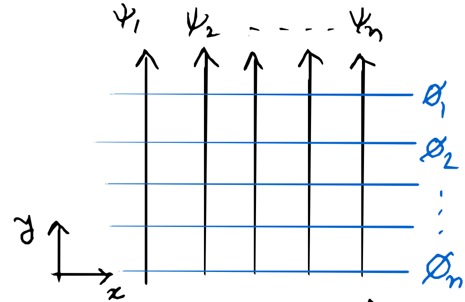
$$\psi = Uy$$

$$\frac{\partial \phi}{\partial x} = U \quad , \quad \frac{\partial \phi}{\partial y} = 0$$

$$\phi = Ux + f(y), \quad \phi = f(x)$$

$$\phi = Ux + \cancel{C}^{\,0}$$

$$\phi = Ux$$



$$\vec{v} = 0\hat{i} + U\hat{j}$$

$$\frac{\partial \psi}{\partial y} = 0 \quad , \quad -\frac{\partial \psi}{\partial x} = U$$

$$\psi = f(x) \qquad \psi = -Ux + f(y)$$

$$\psi = -Ux + \cancel{C}^{\,0}$$

$$\psi = -Ux$$

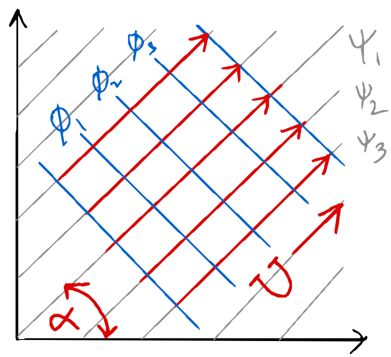
$$\frac{\partial \phi}{\partial x} = 0 \quad , \quad \frac{\partial \phi}{\partial y} = U$$

$$\phi = f(y) \qquad \phi = Uy + f(x)$$

$$\phi = Uy + \cancel{C}^{\,0}$$

$$\phi = Uy.$$

* General uniform flow (inclined by α with x-axis)



$$\vec{v} = U\cos\alpha\,\hat{i} + U\sin\alpha\,\hat{j}$$

$$\Rightarrow \frac{\partial\psi}{\partial y} = U\cos\alpha \quad , \quad -\frac{\partial\psi}{\partial x} = U\sin\alpha$$

$$\Rightarrow \psi = (U\cos\alpha)y + f(x) \qquad \psi = -(U\sin\alpha)x + f(y)$$

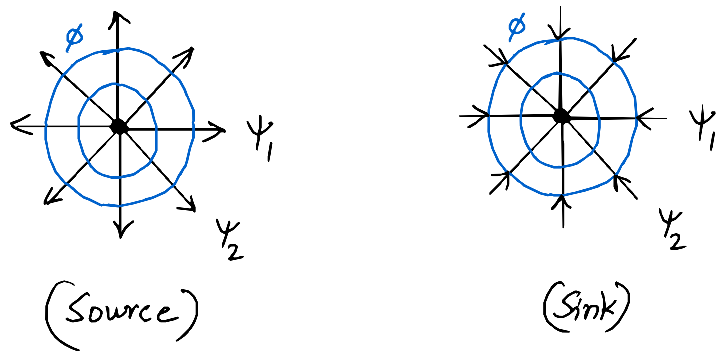
$$\psi = (U\cos\alpha)y - (U\sin\alpha)x + \cancel{c}^{\,0}$$

$$\psi = (U\cos\alpha)y - (U\sin\alpha)x$$

$$\Rightarrow \psi = U(y\cos\alpha - x\sin\alpha)$$

Similarly, $\phi = U(x\cos\alpha + y\sin\alpha)$

② Source and sink :



* We use polar co-ordinate.

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$$

* If the source/sink creates volumetric flow rate of m (per unit length of line)

$$(2\pi r)\,v_r = m$$

$$\Rightarrow v_r = \left(\frac{m}{2\pi r}\right)$$ * radial velocity $v_r \propto 1/r$

Thus, $\vec{v} = \left(\frac{m}{2\pi r}\right)\hat{e}_r + 0\,\hat{e}_\theta$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \quad , \quad -\frac{\partial \psi}{\partial r} = 0$$

$$\Rightarrow \psi = \left(\frac{m}{2\pi}\right)\theta + f(r) \quad , \quad \psi = f(\theta)$$

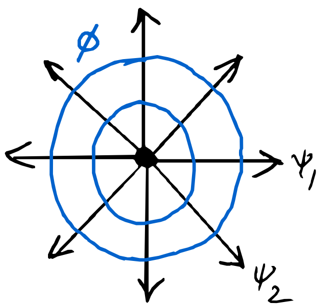
$$\psi = \left(\frac{m}{2\pi}\right)\theta + \cancel{c}^{\,0}$$

$$\Rightarrow \psi = \left(\frac{m}{2\pi}\right)\theta \quad ,$$

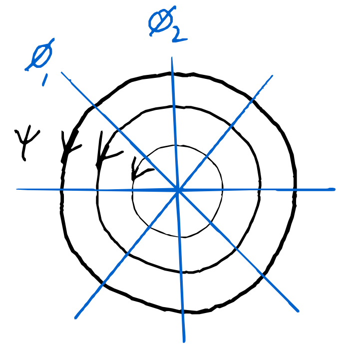
Similarly :

$$\phi = \frac{m}{2\pi}(\ln r)$$

③ Vortex :



(source)



(vortex)

* $\psi = \left(\frac{m}{2\pi}\right)\theta = K\theta$ $\qquad$ $\psi = -K \ln r$

$\phi = \left(\frac{m}{2\pi}\right) \ln r = K \ln r$ $\qquad$ $\phi = K\theta$

(swaping ϕ & ψ)

* Now velocity vector become,

$$\vec{v} = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta = 0\hat{e}_r + \left(\frac{K}{r}\right)\hat{e}_\theta$$

$$\Rightarrow \vec{v} = 0\hat{e}_r + \left(\frac{K}{r}\right)\hat{e}_\theta$$

$\longrightarrow$ check compressible/incompressible flow

$\longrightarrow$ check rotational/irrotational flow

* What does K mean?

$\longrightarrow$ Circulation $\left(K = \frac{\Gamma}{2\pi}\right)$. ← Circulation

$\longrightarrow$ Circulation is defined as closed loop integral of tangential velocity.

$$\Gamma = \oint \vec{v} \cdot d\vec{s}$$

* For vortex flow $\vec{v} = v_\theta \hat{e}_s + 0\,\hat{e}_n$ $\quad [v_s = v_\theta]$

$\therefore \quad \vec{v} \cdot d\vec{s} = v_\theta (r\,d\theta)$

* Close loop integral can be obtained for $0 \leq \theta \leq 2\pi$

* Thus, $\Gamma = \int_0^{2\pi} v_\theta\, r\, d\theta = \int_0^{2\pi} (K/r)\, r\, d\theta = 2\pi K$

* Now we can express the velocity vector using the circulation Γ as

$$\vec{v} = (0)\,\hat{e}_r + \left(\frac{\Gamma}{2\pi r}\right)\hat{e}_\theta$$

$$\left.\begin{aligned} \psi &= -\left(\frac{\Gamma}{2\pi}\right)\ln r \\ \phi &= \left(\frac{\Gamma}{2\pi}\right)\theta \end{aligned}\right\} \text{free vortex (irrotational)}$$

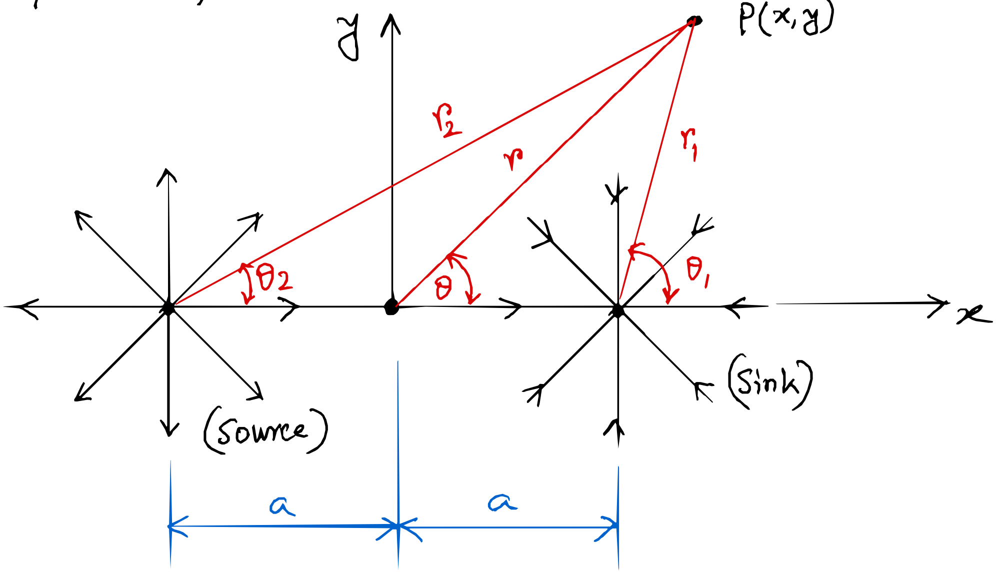
Flow	Velocity	ψ	ϕ
Uniform flow.	$\vec{v} = u\hat{i} + v\hat{j}$ $u = U\cos\alpha$ $v = U\sin\alpha$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$\phi = U(x\cos\alpha + y\sin\alpha)$
Source Sink	$\vec{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta$ $v_r = \frac{m}{2\pi r}$ $v_\theta = 0$	$\psi = \left(\frac{m}{2\pi}\right)\theta$	$\phi = \left(\frac{m}{2\pi}\right)\ln r$
Vortex	$\vec{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta$ $v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi r}$	$\psi = -\left(\frac{\Gamma}{2\pi}\right)\ln r$	$\phi = \left(\frac{\Gamma}{2\pi}\right)\theta$

* Flow field visualization website

Go to: potentialflow.com

Doublet

* Source and sink of same strength (m) in close proximity.



* For source, $\psi_1 = \left(\frac{m}{2\pi}\right)\theta_2$

  For sink, $\psi_2 = \left(-\frac{m}{2\pi}\right)\theta_1$

  ψ is a scalar !!

* Combined stream function,

$$\psi = \psi_1 + \psi_2 = -\left(\frac{m}{2\pi}\right)(\theta_1 - \theta_2)$$

* Taking tangent on both side

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2)$$

* Trigonometric identity, $\tan(A-B) = \left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)$ gives

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1 \tan\theta_2}$$

* From the triangles in figure

$$\tan\theta_1 = \left(\frac{r\sin\theta}{r\cos\theta - a}\right) \qquad \tan\theta_2 = \left(\frac{r\sin\theta}{r\cos\theta + a}\right)$$

* Thus, $$\tan(\theta_1 - \theta_2) = \frac{\left(\frac{r\sin\theta}{r\cos\theta - a}\right) - \left(\frac{r\sin\theta}{r\cos\theta + a}\right)}{1+\left(\frac{r^2\sin^2\theta}{r^2\cos^2\theta - a^2}\right)}$$

$$= \left(\frac{r^2\cancel{\sin\theta\cos\theta} + ar\sin\theta - r^2\cancel{\sin\theta\cos\theta} + ar\sin\theta}{r^2\cos^2\theta - a^2 + r^2\sin^2\theta}\right)$$

$$= \left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

* Thus, $\tan\left(-\frac{2\pi\psi}{m}\right) = \left(\frac{2ar\sin\theta}{r^2-a^2}\right)$

$$\Rightarrow \psi = -\left(\frac{m}{2\pi}\right)\tan^{-1}\left\{\frac{2ar\sin\theta}{r^2-a^2}\right\}$$ (Source + Sink pair)

* If a is small, $\psi = -\left(\frac{m}{2\pi}\right)\frac{2ar\sin\theta}{r^2-a^2}$

$$\Rightarrow \psi = -\left\{\frac{mar\sin\theta}{\pi(r^2-a^2)}\right\}$$

* Observations (*) for $a \to 0$, $m \to \infty$

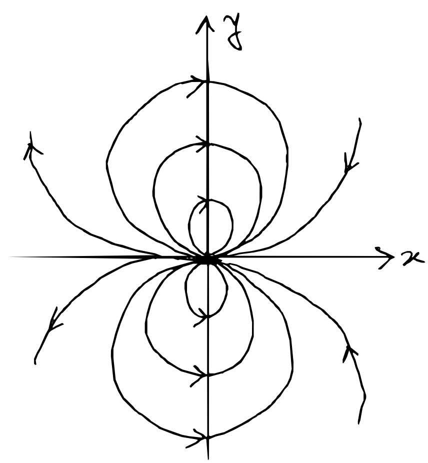
small $a \longleftrightarrow$ large m

(am/π remains constant)

* Thus, $\psi = -\left(\frac{K}{r}\right)\sin\theta$

Here, K = Doublet strength

(Determined by strength of the source & sink as well as distance between them)



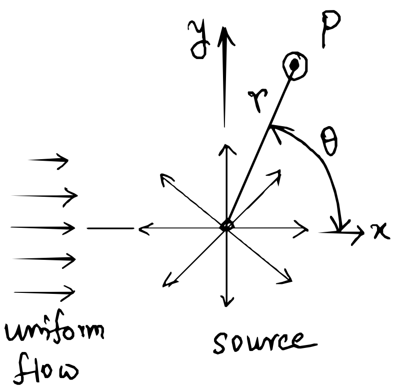
* Similarly it can be shown that

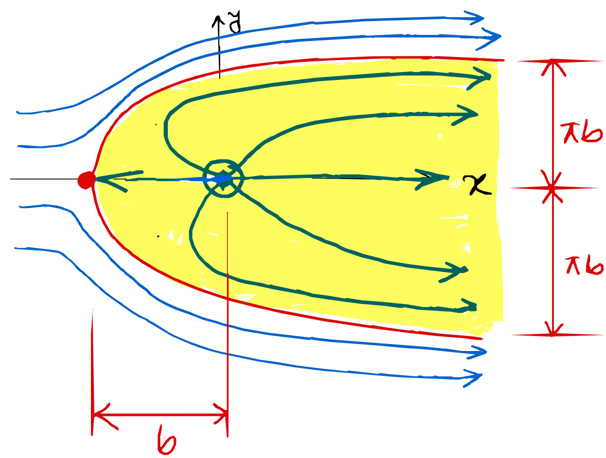
$\phi = \left(\frac{K}{r}\right)\cos\theta$

Do it @ home

Superposition of basic flows

① Half body (uniform flow + source)





$*$ $\psi = U r \sin\theta + \left(\frac{m}{2\pi}\right)\theta$ (Simple)

$*$ Stagnation point: $U = v_r \Rightarrow U = \left(\frac{m}{2\pi b}\right)$

Thus, $b = \left(\frac{m}{2\pi U}\right)$

$*$ $\psi_{stagnation} = \psi(r = b \text{ and } \theta = \pi)$

$\psi_{stagnation} = U b \sin(\pi) + \frac{m}{2\pi} \cdot \pi = \left(\frac{m}{2}\right)$

* The equation for the streamline passing through the stagnation point.

$$\left(\frac{m}{2}\right) = Ur\sin\theta + \left(\frac{m}{2\pi}\right)\theta$$

$$\Rightarrow \pi b U = Ur\sin\theta + bU\theta \quad \left(\text{put } b = \frac{m}{2\pi U}\right)$$

$$\text{or } \frac{m}{2} = \pi b U$$

* Solving for r gives,

$$r = \left\{\frac{b(\pi-\theta)}{\sin\theta}\right\} \quad \text{or, } y = b(\pi-\theta)$$

* From the stream function we can obtain the velocity components at any point.

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

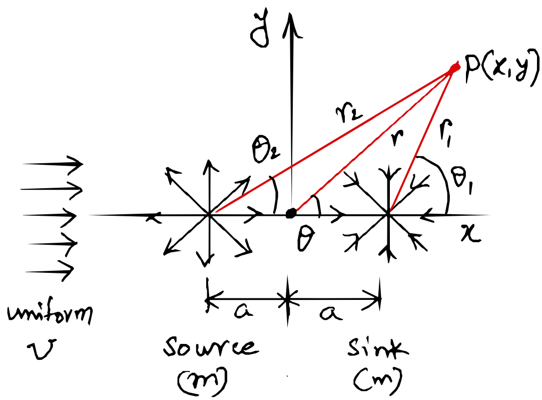
Thus, $\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$

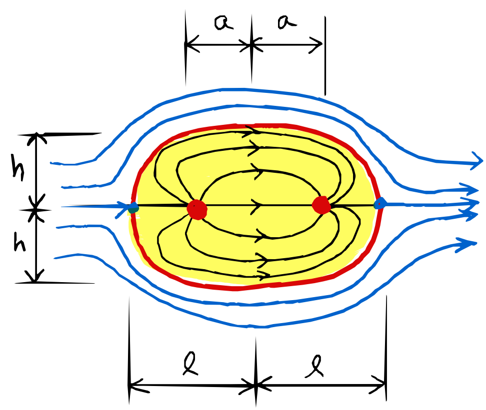
$$v^2 = v_r^2 + v_\theta^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

$$\left[\text{Since, } b = \frac{m}{2\pi U}\right]$$

* From $v^2(r,\theta)$ known, we can apply the Bernoulli's equation between any two points (flow is incompressible & irrotational + inviscid) to obtain pressure variations. (cool)

② Rankine Oval (Uniform flow + Source + sink)





* Stream function

$$\psi = \psi_{uniform} + \psi_{source} + \psi_{sink}$$ (scalar addition)

$$\Rightarrow \psi = Ur\sin\theta + \left(\frac{m}{2\pi}\right)\theta_2 - \left(\frac{m}{2\pi}\right)\theta_1$$

$$\Rightarrow \psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2)$$

$$\Rightarrow \psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

(See doublet flow)

* In cartesian co-ordinate $\begin{cases} x = r\cos\theta, \ y = r\sin\theta \\ r^2 = x^2+y^2 \end{cases}$

$$\psi = Uy - \left(\frac{m}{2\pi}\right)\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right)$$

* when $\psi = 0$, it is an oval (Not an ellipse)

* Obtaining the dimension l and h

* To find l we need to determine the stagnation points ($\vec{v}=0$) along x-axis ($y=0$)

$\vec{v} = v_x\hat{i} + v_y\hat{j}$ (cartesian co-ordinate)

$$v_x = \frac{\partial\psi}{\partial y} = U - \left(\frac{m}{2\pi}\right)\left[\frac{1}{1+\left(\frac{2ay}{x^2+y^2-a^2}\right)^2}\right]\left[\frac{(x^2+y^2-a^2)2a - 4ay^2}{(x^2+y^2-a^2)^2}\right]$$

$$\Rightarrow v_x = U - \left(\frac{m}{2\pi}\right)\frac{\cancel{(x^2+y^2-a^2)^2}}{(x^2+y^2-a^2)^2+4a^2y^2} \times \frac{2a(x^2+y^2-a^2)-4ay^2}{\cancel{(x^2+y^2-a^2)^2}}$$

$$\Rightarrow v_x = U - \left(\frac{m}{\cancel{2}\pi}\right)\left[\frac{\cancel{2}a(l^2-a^2)}{(l^2-a^2)^2}\right] \quad \left(\begin{matrix}\text{Putting } x = l \\ \text{and } y = 0\end{matrix}\right)$$

$$\Rightarrow v_x = U - \frac{ma}{\pi(l^2-a^2)}$$

$$v_y = -\frac{\partial\psi}{\partial x} = 0 + \left(\frac{m}{2\pi}\right)\left[\frac{1}{1+\left(\frac{2ay}{x^2+y^2-a^2}\right)^2}\right]\left[\frac{(x^2+y^2-a^2)\cdot 0 - 4axy}{(x^2+y^2-a^2)^2}\right]$$

$$\Rightarrow v_y = \frac{m}{2\pi}\frac{\cancel{(x^2+y^2-a^2)^2}}{(x^2+y^2-a^2)^2+4a^2y^2} \times (-)\frac{4axy}{\cancel{(x^2+y^2-a^2)^2}}$$

$$\Rightarrow v_y = -\left(\frac{m}{2\pi}\right)\frac{4axy}{(x^2+y^2-a^2)^2+4a^2y^2} = 0 \quad (y=0)$$

Thus @ stagnation point $v_x = 0$

$$\Rightarrow U - \frac{ma}{\pi(\ell^2 - a^2)} = 0 \quad \Rightarrow (\ell^2 - a^2) = \frac{ma}{\pi U}$$

$$\Rightarrow \ell^2 = \left(a^2 + \frac{ma}{\pi U}\right)$$

$$\Rightarrow \left(\frac{\ell}{a}\right)^2 = \left(1 + \frac{m}{\pi a U}\right)$$

$$\Rightarrow (\ell/a) = \sqrt{1 + \left(\frac{m}{\pi a U}\right)}$$

* To obtain h we determine the location of y-axis and $\psi = 0$ lines,

Putting $y = h$ and $\psi = 0$ in the ψ expression along with $x = 0$, we get

$$0 = Uh - \left(\frac{m}{2\pi}\right) \tan^{-1}\left[\frac{2ah}{0^2 + h^2 - a^2}\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2ah}{h^2 - a^2}\right) = \left(\frac{Uh}{m}\right) 2\pi$$

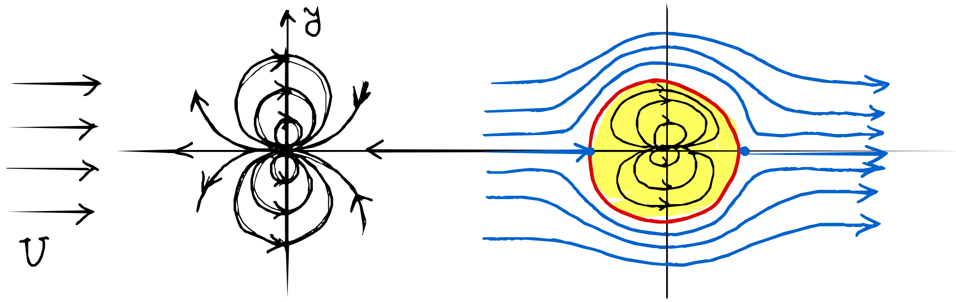
$$\Rightarrow \left(\frac{2ah}{h^2 - a^2}\right) = \tan\left(\frac{2\pi U h}{m}\right)$$

$$\Rightarrow h = \left(\frac{h^2 - a^2}{2a}\right) \tan\left(\frac{2\pi U h}{m}\right)$$

(implicit expression)

* requires trial & error to solve such equation.

③ Flow around a circular cylinder
(Uniform flow + Doublet)



$$\psi = \psi_{uniform} + \psi_{doublet}$$

$$\Rightarrow \psi = Ur\sin\theta - \left(\frac{K}{r}\right)\sin\theta$$ (see doublet flow)

$$\Rightarrow \psi = \left(U - \frac{K}{r^2}\right) r\sin\theta$$

* For flow around a cylinder it is important (necessary) to have $\psi = 0$ at $r = a$
(a = cylinder radius)

putting $\psi = 0$ and $r = a$ we obtain that

$$K = Ua^2$$

Remember it is a choice. we need to select proper value of K to simulate flow over a cylinder with known radius and free stream velocity.

* putting $K = Ua^2$ gives,

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

* Since ψ is known we can obtain the velocity field as

$$\vec{v} = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\hat{e}_r - \left(\frac{\partial\psi}{\partial r}\right)\hat{e}_\theta$$

$$\Rightarrow \vec{v} = \frac{1}{r}\cdot Ur\left(1 - \frac{a^2}{r^2}\right)\cos\theta\ \hat{e}_r - U\sin\theta\left[1 + \frac{a^2}{r^2}\right]\hat{e}_\theta$$

$$\Rightarrow \vec{v} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\ \hat{e}_r - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\ \hat{e}_\theta$$

* Thus, $v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$

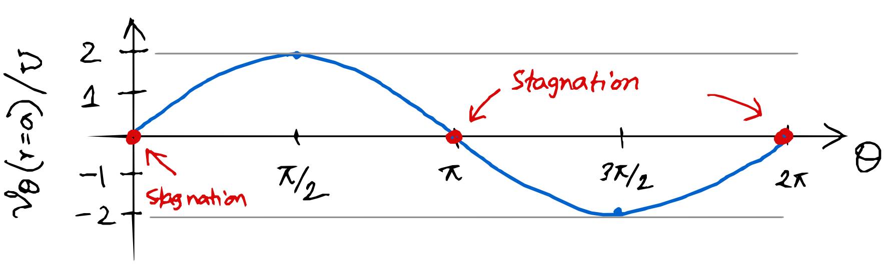
$$v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

* what is the velocity variation on the cylinder surface?

$$v_r(r=a) = 0$$

$$v_\theta(r=a) = -2U\sin\theta$$

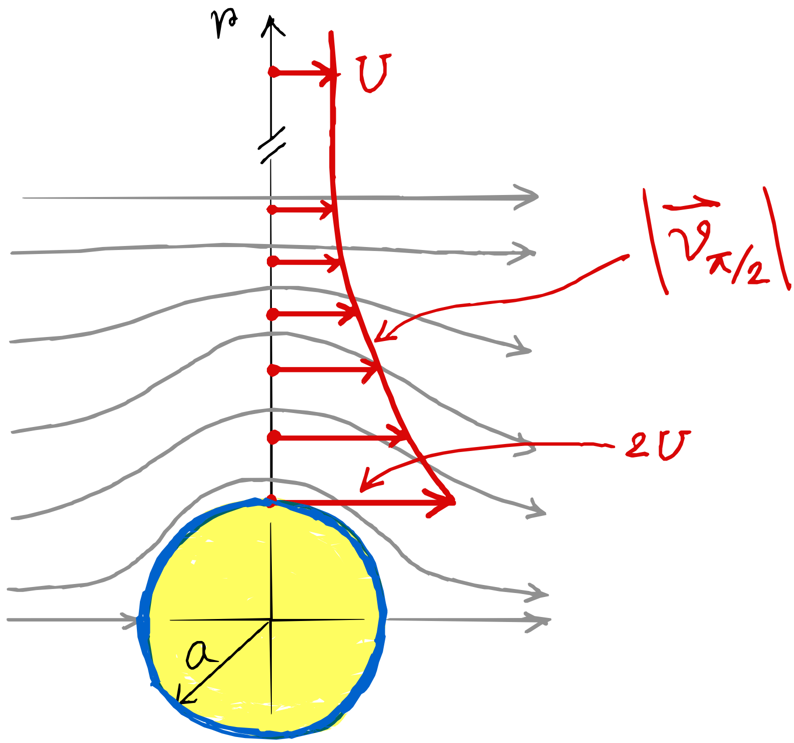
$$\boxed{\vec{v}_{surf} = -2U\sin\theta\ \hat{e}_\theta}$$



* How does velocity varies along y-axis?

Putting $\theta = \pi/2$ we get

$$\vec{v}_{\pi/2} = -U\left(1 + \frac{a^2}{r^2}\right)\hat{e}_\theta$$



* why there is a non-zero velocity at the cylinder surface?

* what happened to no-slip boundary condition?

* How does pressure varies on the cylinder surface?

* Applying Bernoulli's equation between two points (one of them is upstream far field and one of them is on the cylinder surface) we get

$$P_\infty + \frac{1}{2}\rho v_\infty^2 = P_s + \frac{1}{2}\rho v_s^2$$

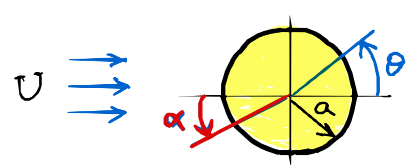
We know, $v_s^2 = (-2U\sin\theta)^2 = 4U^2\sin^2\theta$

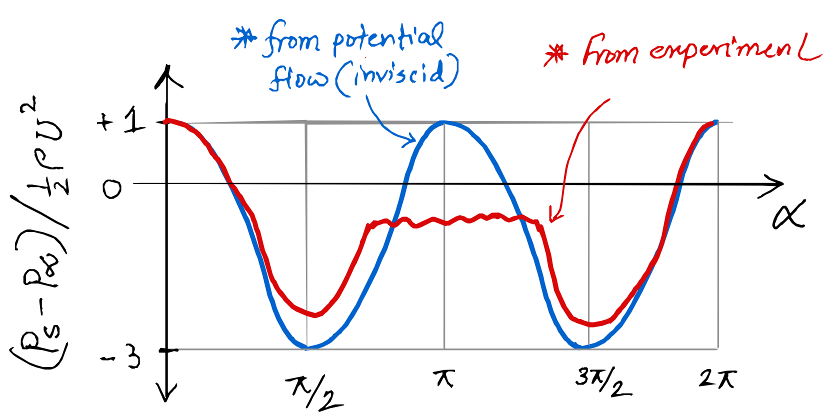
Thus, $P_s = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(4U^2\sin^2\theta)$

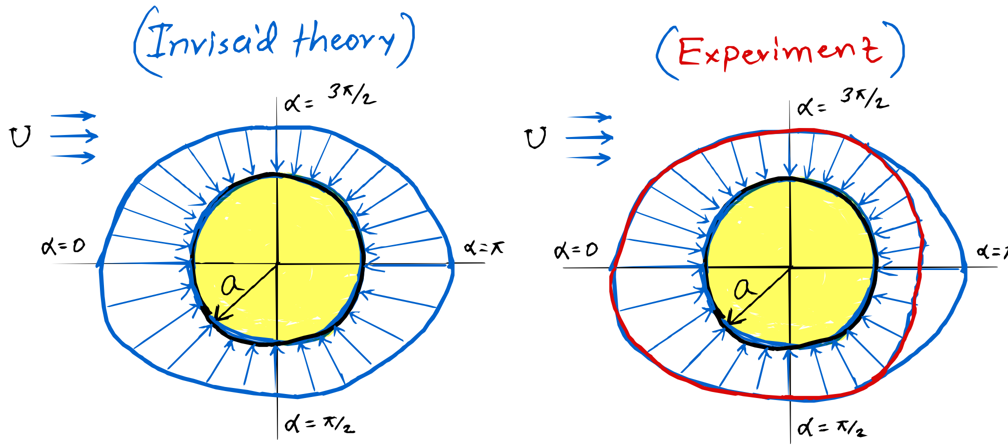
$$\Rightarrow P_s = P_\infty + \frac{1}{2}\rho U^2\left[1 - 4\sin^2\theta\right]$$

$$\Rightarrow \left(\frac{P_s - P_\infty}{\frac{1}{2}\rho U^2}\right) = 1 - 4\sin^2(\pi+\theta) = 1 - 4\sin^2\alpha$$

(This is known as Pressure coefficient C_p)







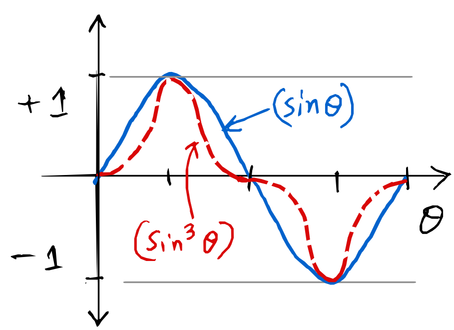
* From surface pressure $P_s = P_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta)$ we can calculate the vertical (Lift) and horizontal (Drag) force.

$$F_y = -\int_0^{2\pi} P_s \sin\theta \; a\,d\theta = a\int_0^{2\pi} P_s \sin\theta \, d\theta$$

$$= aP_\infty \int_0^{2\pi} \sin\theta \, d\theta + \frac{a\rho U^2}{2}\int_0^{2\pi} \sin\theta \, d\theta - 2a\rho U^2 \int \sin^3\theta \, d\theta$$

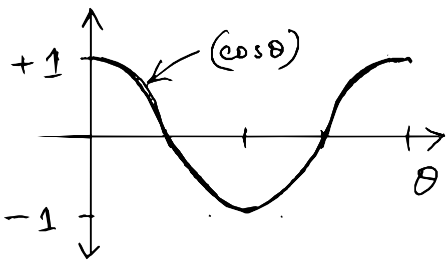
$\therefore F_y = 0$

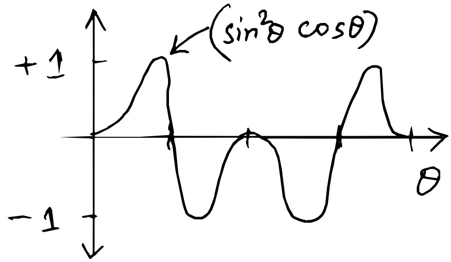
(No surprise here) expected



$$* \; F_x = -\int_0^{2\pi} P_s \cos\theta \, a \, d\theta$$

$$= -aP_s \int_0^{2\pi} \cos\theta \, d\theta - \frac{a\rho U}{2} \int_0^{2\pi} \cos\theta \, d\theta + 2aU \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta$$





$\therefore F_x = 0$ (surprised !!)

* Think about viscosity. **

→ d'Alembert's paradox (1752)

→ Prandtl explained this paradox in (1904)

* What will happen if we add a free vortex (in other word circulation) in a flow over cylinder

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta - \left(\frac{\Gamma}{2\pi}\right)\ln(r)$$

* From ψ we can obtain velocity vector

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$$

$$\Rightarrow \vec{v} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{e}_r - \frac{\partial \psi}{\partial r}\,\hat{e}_\theta$$

$$\Rightarrow \vec{v} = \left[\frac{1}{r}\,Ur\left(1 - \frac{a^2}{r^2}\right)\cos\theta\right]\hat{e}_r - \left[U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right]\hat{e}_\theta$$

$$\Rightarrow \vec{v} = \left[U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\right]\hat{e}_r - \left[U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right]\hat{e}_\theta$$

* Stagnation points can be obtained by setting

$$\vec{v} = 0 \quad \text{or} \quad v^2 = 0 \Rightarrow v_r^2 + v_\theta^2 = 0$$

* We would expect the stagnation points to be on the cylinder surface ($r = a$)

$$v_r^{surf} = 0, \quad v_\theta^{surf} = -2U\sin\theta + \frac{\Gamma}{2\pi a}$$

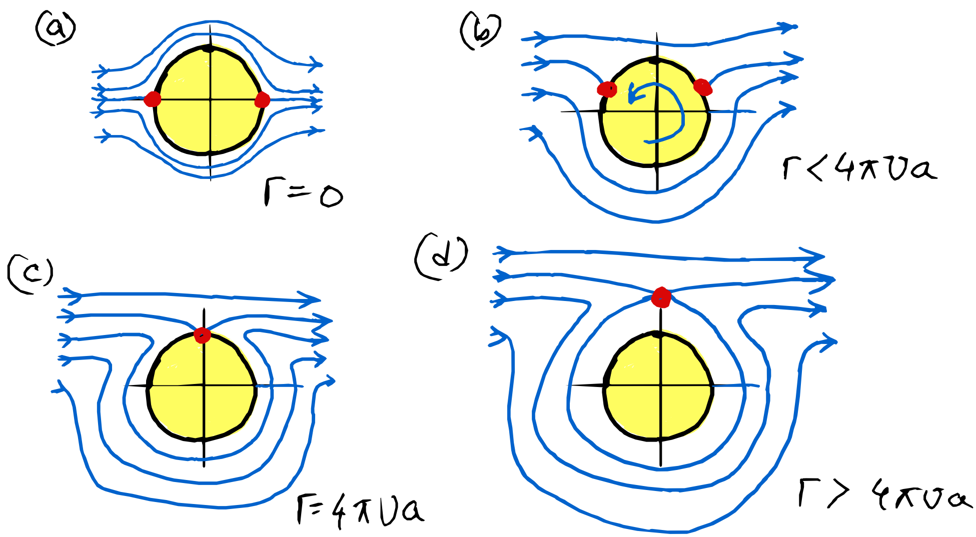
Thus, $v^2 = 0$ becomes,

$$\Rightarrow \sin(\theta_{stag}) = \left(\frac{\Gamma}{4\pi U a}\right)$$

* We see that, when $\Gamma \leqslant 4\pi U a$ the above expression is valid, indicating stagnation point(s) at $r = a$.

* For, $\Gamma > 4\pi U a$ the stagnation point is not on the cylinder surface.

* Thus 4 different flow characteristics can be obtained.



* For $\Gamma > 4\pi U a$ the location of stagnation point can be obtained by setting $\theta = \pi/2$,

$$v_r^{\theta=\pi/2} = 0, \qquad v_\theta^{\theta=\pi/2} = -U\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r}$$

Thus, $v^2 = 0$ becomes,

$$U + \frac{Ua^2}{r^2} = \frac{\Gamma}{2\pi r} \Rightarrow Ur^2 + (Ua^2) = \left(\frac{\Gamma}{2\pi}\right) r$$

$$\Rightarrow (2\pi U) r^2 - (\Gamma) r + (2\pi U a^2) = 0$$

* The above quadratic expression can be solved

$$r_{stag} = \left(\frac{\Gamma \pm \sqrt{\Gamma^2 - 16\pi^2 U^2 a^2}}{(4\pi U)} \right)$$

$$\Rightarrow r_{stag} = \left(\frac{\Gamma}{4\pi U}\right) \pm (4\pi U a) \sqrt{\left(\frac{\Gamma}{4\pi U a}\right)^2 - 1}$$

(we can have multiple stagnation points)

* Since the velocity variation along cylinder surface $\vec{v}_s$ is known as,

$$\vec{v}_s = 0\,\hat{e}_r - \left(2U\sin\theta - \frac{\Gamma}{2\pi a}\right)\hat{e}_\theta$$

we can obtain the expression for pressure variation along the cylinder surface as well.

* $$P_s + \frac{1}{2}\rho v_s^2 = P_\infty + \frac{1}{2}\rho v_\infty^2$$

$$\Rightarrow P_s = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left[-2U\sin\theta + \frac{\Gamma}{2\pi a}\right]^2$$

$$\Rightarrow P_s = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left[4U^2\sin^2\theta - \left(\frac{2U\Gamma}{\pi a}\right)\sin\theta + \frac{\Gamma^2}{4\pi^2 a^2}\right]$$

$$\Rightarrow P_s = P_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho U^2\left[4\sin^2\theta - \left(\frac{2\Gamma}{\pi U a}\right)\sin\theta + \frac{\Gamma^2}{4\pi^2 U^2 a^2}\right]$$

$$\Rightarrow P_s = P_\infty + \frac{1}{2}\rho U^2\left[1 - 4\sin^2\theta + \left(\frac{2\Gamma}{4Ua}\right)\sin\theta - \frac{\Gamma^2}{4\pi^2 U^2 a^2}\right]$$

* Similarly to non-lifting flow, the lift & drag force here can be obtained by integrating the pressure distribution on the surface.

$$F_x = \int_0^{2\pi} P_s \cos\theta \, a \, d\theta = 0 \quad \text{[Do it @ home]}$$

$$F_y = -\int_0^{2\pi} P_s \sin\theta \, a \, d\theta \neq 0$$

$$\Rightarrow F_y = -a \int_0^{2\pi} \left[P_\infty + \frac{1}{2}\rho U^2 \left\{ 1 - 4\sin^2\theta + \left(\frac{2\Gamma}{\pi U a}\right)\sin\theta - \frac{\Gamma^2}{4\pi^2 U^2 a^2} \right\} \right] \sin\theta \, d\theta$$

$$\Rightarrow F_y = -a P_\infty \int_0^{2\pi} \sin\theta \, d\theta - \frac{1}{2}\rho a U^2 \int_0^{2\pi} \sin\theta \, d\theta - \left(\frac{\rho \Gamma U}{\pi}\right) \int_0^{2\pi} \sin^2\theta \, d\theta + \left(\frac{\rho \Gamma^2}{8\pi^2}\right) \int_0^{2\pi} \sin\theta \, d\theta$$

$$\Rightarrow F_y = -\left(\frac{\rho \Gamma U}{\pi}\right) \int_0^{2\pi} \sin^2\theta \, d\theta = -\rho \Gamma U$$

(Magnus effect)

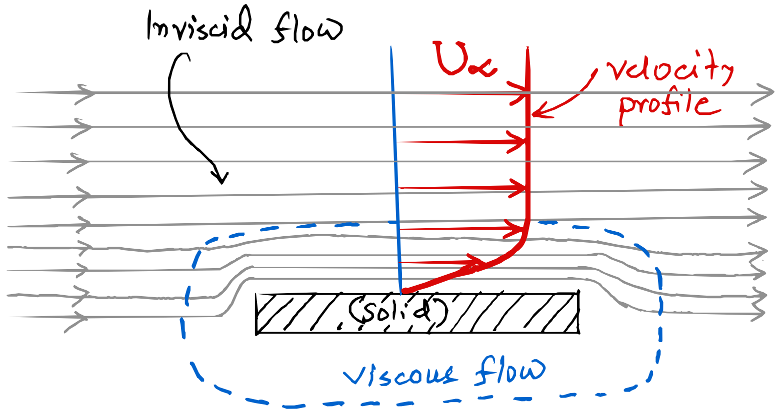
* Lift Force per unit span, $\mathcal{L} = -\rho U \Gamma$

(Kutta-Joukowski Theorem)

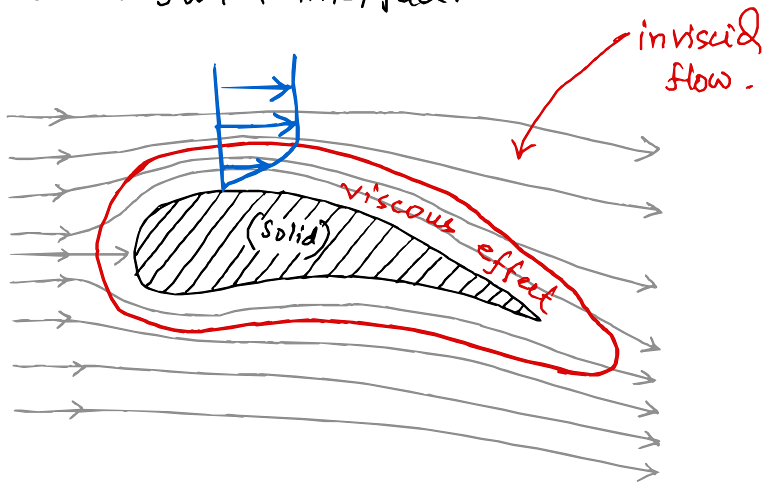
* Drag force per unit span, $D = 0$

(d'Alembert's Paradox)

* Prandtl (1904) showed that, the drag is the manifestation of the thin boundary layer formation due to viscous effect.



* Usually for most application's viscous region is small and it is very thin around the solid-fluid interface.



Viscous Force

* Viscous force can be obtained from viscous shear stress, $\overline{\overline{\tau}} = \mu \nabla \vec{v}$ (Newton)

  $\overline{\overline{\tau}}$ → stress tension

* Viscous shear stress $\overline{\overline{\tau}}$ and pressure are both surface force. Thus their spatial variation gives the force per unit volume

$$\vec{F}_{pressure} = -\nabla P \quad \text{(Gradient)}$$

$$\vec{F}_{viscous} = \nabla \cdot \overline{\overline{\tau}} \quad \left[\begin{array}{l}\text{Divergence !!} \\ \text{No negative sign}\end{array}\right]$$

* Force balance $\Sigma \vec{F} = m\vec{a}$

$$\Rightarrow \underbrace{-\nabla P}_{\text{(Pressure force)}} + \underbrace{\rho \vec{g}}_{\text{(Gravity force)}} + \underbrace{\nabla \cdot \overline{\overline{\tau}}}_{\text{(Viscous force)}} = \underbrace{\rho \frac{D\vec{v}}{Dt}}_{\text{(momentum change (inertia))}}$$

$$\Rightarrow -\nabla P + \rho \vec{g} + \nabla \cdot \underbrace{(\mu \nabla \vec{v})}_{\text{Newtonian fluid}} = \rho \left(\frac{D\vec{v}}{Dt}\right)$$

both surface force

$$\Rightarrow -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{v} = \rho \frac{D\vec{v}}{Dt}$$

$$\Rightarrow \rho \frac{D\vec{v}}{Dt} = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{v}$$

(Navier-Stokes equation)

$$\Rightarrow \boxed{\underbrace{\rho \frac{\partial \vec{v}}{\partial t}}_{\text{(Transient term)}} + \underbrace{\left(\rho\, \vec{v} \cdot \nabla \vec{v}\right)}_{\text{(Convection term)}} = \underbrace{-\nabla P}_{\text{(Pressure term)}} + \underbrace{\rho \vec{g}}_{\text{(Gravity term)}} + \underbrace{\mu \nabla^2 \vec{v}}_{\text{(Diffusion term)}}}$$

* There are multiple ways we can prove this equation and represent it.

* Try to be comfortable with any form and understand it. For proof read section 4.3 from White's book.

Some comments on N-S equation

1. N-S equation is non-linear due to the presence of Convection term $(\rho \vec{v} \cdot \nabla \vec{v})$.

2. N-S does not have an exact solution (Million-Dollar Question)

3. See video-19 on youtube.

4. Though general solution is not available case specific analytical solutions are obtained

Detail in class

* Coutte Flow (Moving plate)
* Poiseulle Flow (Parallel plates & tubes)
* Annular Flow (Tube annulus)
* Stokes Flow (Creeping flow)
* Hele-shaw Flow

Components of N-S equation

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v}\cdot\nabla\vec{v} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{v}$$

Transient term: $\left(\rho\frac{\partial u}{\partial t}\right)\hat{i} + \left(\rho\frac{\partial v}{\partial t}\right)\hat{j} + \left(\rho\frac{\partial w}{\partial t}\right)\hat{k}$

convective term: $\rho\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\vec{v}$

$$= \rho\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\right]$$

$$= \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\hat{i}$$
$$+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\hat{j}$$
$$+\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\hat{k}$$

Pressure term: $-\left(\frac{\partial P}{\partial x}\right)\hat{i} - \left(\frac{\partial P}{\partial y}\right)\hat{j} - \left(\frac{\partial P}{\partial z}\right)\hat{k}$

gravity term: $(\rho g_x)\hat{i} + (\rho g_y)\hat{j} + (\rho g_z)k$

Diffusion term: $\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\hat{i}$

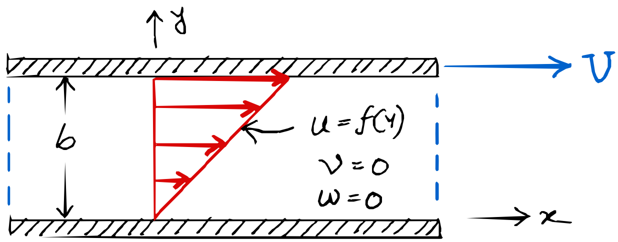
$$+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\hat{j}$$
$$+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\hat{k}$$

x	$\rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$
y	$\rho\frac{\partial v}{\partial t}+\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$
z	$\rho\frac{\partial w}{\partial t}+\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$

* Derive components of N-S equation in cylindrical coordinate.

Viscous Flow

* Couette Flow: plate driven flow (steady)



* x-component of N-S:

$$\rho\underbrace{\cancel{\frac{\partial u_x}{\partial t}}}_{0\text{ (steady)}} + \rho\left(u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{u \neq f(x)} + \overbrace{\cancel{v}}^{v=0}\frac{\partial u}{\partial y} + w\underbrace{\cancel{\frac{\partial u}{\partial z}}}_{u \neq f(z)}\right) = -\frac{\partial P}{\partial x} + \rho\underbrace{\cancel{g_x}}_{g_x=0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{u \neq f(x)} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{u \neq f(z)}\right)$$

* Reduced x-component of N-S

$$0 = -\left(\frac{\partial P}{\partial x}\right) + \mu\frac{\partial^2 u}{\partial y^2}$$

* If $\left(\frac{\partial P}{\partial x}\right)$ = constant (pressure varies linearly in x direction)

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\left(\frac{\partial P}{\partial x}\right)$$

(integrating 2 times with respect to y gives)

$$u = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)y^2 + \underbrace{C_1 y + C_2}_{\text{constant}}$$

* From boundary conditions:

(a) $y=0, \quad u=0 \longrightarrow 0 = C_2$

(b) $y=b, \quad u=U \longrightarrow U = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)b^2 + C_1 b$

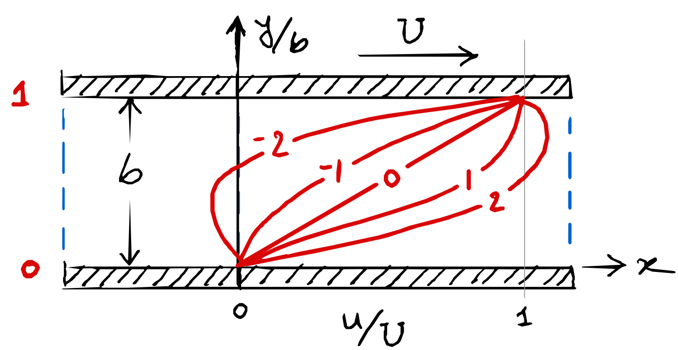
$$\Rightarrow C_1 = \left(\frac{U}{b}\right) - \frac{b^2}{2\mu b}\left(\frac{\partial P}{\partial x}\right)$$

$$\Rightarrow C_1 = \left(\frac{U}{b}\right) - \frac{b}{2\mu}\left(\frac{\partial P}{\partial x}\right)$$

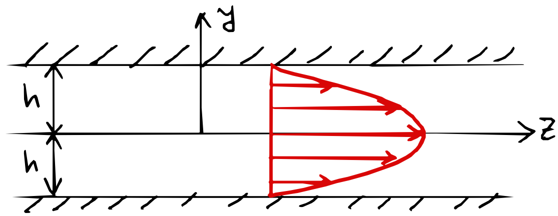
Thus, $u = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)y^2 + \left(\frac{Uy}{b}\right) - \frac{b}{2\mu}\left(\frac{\partial P}{\partial x}\right)y$

$$\Rightarrow u = U\left(\frac{y}{b}\right) + \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)\left(y^2 - by\right)$$

$$* \quad \left(\frac{u}{U}\right) = \left(\frac{y}{b}\right) + \frac{b^2}{2\mu}\left(\frac{\partial P}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \left(\frac{y}{b}\right)\right]$$



* Poiseulle Flow (Parallel plates)



$$\begin{cases} u = f(y) \\ v = 0 \\ w = 0 \end{cases}$$

x-component of N-S:

$$0 = -\frac{\partial P}{\partial z} + \mu \frac{\partial^2 u}{\partial y^2}$$

Solution: $u = \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) y^2 + C_1 y + C_2$

From $u(y = \pm h) = 0$, $\begin{cases} 0 = \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) h^2 + C_1 h + C_2 \\ 0 = \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) h^2 - C_1 h + C_2 \end{cases}$

$(-)$ $C_1 = 0$, $(+)$ $0 = \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) h^2 + C_2$

$$\Rightarrow C_2 = -\frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) h^2$$

Thus, $u = \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) y^2 - \frac{1}{2\mu}\left(\frac{\partial P}{\partial z}\right) h^2$

$$\Rightarrow u = \frac{h^2}{2\mu}\left(-\frac{\partial P}{\partial z}\right)\left[1 - \left(\frac{y}{h}\right)^2\right]$$

(2-D parabola)

* $u_{max} = u(y=0) = \frac{h^2}{2\mu}\left(-\frac{\partial P}{\partial z}\right)$

* what would be the flow rate?

$$q = \int \vec{v} \cdot d\vec{A}$$

Here, $\vec{v} = u\,\hat{\imath} = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\left[1-\left(\frac{y}{h}\right)^2\right]\hat{\imath}$

$d\vec{A} = dy\,\hat{\imath}$ (unit depth)

$$\therefore q = \int_{-h}^{h} \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\left[1-\left(y/h\right)^2\right] dy$$

$$\Rightarrow q = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\left[y - \frac{1}{h^2}\cdot\frac{y^3}{3}\right]_{-h}^{h}$$

$$\Rightarrow q = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\left[2h - \frac{2h}{3}\right]$$

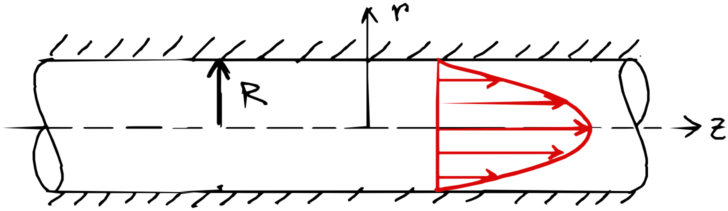
$$\Rightarrow \underbrace{q}_{\text{effect}} = \left(\frac{2h^3}{3\mu}\right)\underbrace{\left(-\frac{\partial p}{\partial z}\right)}_{\text{cause}}$$

coefficient relating cause & effect

* average velocity:

$$V_{avg}\cdot(2h) = q \Rightarrow V_{avg} = \frac{h^2}{3\mu}\left(-\frac{\partial p}{\partial z}\right) = \left(\frac{3}{2}\right) u_{max}$$

** Poiseulle Flow (Circular tube)



$$\begin{cases} v_z = f(r) \\ v_r = 0 \\ v_\theta = 0 \end{cases}$$

z-component of N-S:

$$\rho \cancelto{0}{\frac{\partial v_z}{\partial t}} + \rho\left(\cancelto{0}{v_r \frac{\partial v_z}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancelto{0}{v_z \frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \cancelto{0}{\rho g_z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right]$$

$$\Rightarrow -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0$$

* Solution can be obtained by integrating 2 times considering $\left(-\frac{\partial P}{\partial z}\right)$ = Constant,

$$v_z = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) r^2 + C_1 \ln r + C_2$$

* Boundary conditions: at $r = R$, $v_z = 0$

at $r = 0$, v_z = finite

$\therefore$ $C_1 = 0$ (otherwise v_z would be $-\infty$)

and, $0 = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) R^2 + C_2 \Rightarrow C_2 = -\frac{1}{4\mu}\left(\frac{\partial P}{\partial x}\right) R^2$

* Thus, $$v_z = \frac{R^2}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[1 - \left(r/R\right)^2\right]$$

(3-D Parabola)

* v_{max} is obtained at $r=0$: $v_{max} = \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial z}\right)$

* Flow rate, $q = \int \vec{v} \cdot d\vec{A}$

$$\Rightarrow q = \int_0^R \frac{R^2}{4\mu}\left(-\frac{\partial p}{\partial z}\right)\left[1-\left(r/R\right)^2\right] \hat{e}_z \cdot (2\pi r\, dr)\, \hat{e}_z$$

$$\Rightarrow q = \frac{\pi R^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$\Rightarrow q = \frac{\pi R^2}{2\mu}\left(-\frac{\partial p}{\partial z}\right)\left[\frac{R^2}{2} - \frac{1}{R^2}\cdot\frac{R^4}{4}\right]$$

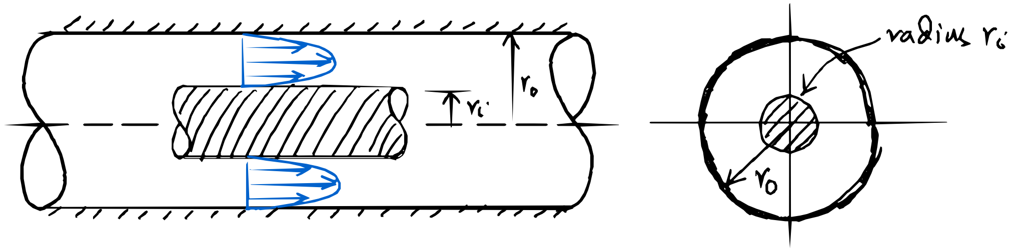
$$\Rightarrow \underbrace{q}_{\text{effect}} = \frac{\pi R^4}{8\mu}\underbrace{\left(-\frac{\partial p}{\partial z}\right)}_{\text{cause}}$$

coefficient relating cause & effect

* Average velocity,

$$V_{avg} \cdot \pi R^2 = q \Rightarrow V_{avg} = \frac{R^2}{8\mu}\left(-\frac{\partial p}{\partial z}\right) = 2\, V_{max}$$

* Annular Flow (circular annulus)



* z-component of N-S equation:

$$0 = -\frac{\partial P}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial v_z}{\partial r}\right)\right\}$$

* General solution, $v_z = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) r^2 + C_1 \ln r + C_2$

* Boundary conditions

(a) $r = r_i$, $v_z = 0 \rightarrow 0 = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) r_i^2 + C_1 \ln r_i + C_2$

(b) $r = r_o$, $v_z = 0 \rightarrow 0 = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) r_o^2 + C_1 \ln r_o + C_2$

(−) $0 = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right)\left(r_o^2 - r_i^2\right) + C_1 \ln\left(\frac{r_o}{r_i}\right) + 0$

$$\Rightarrow C_1 = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\frac{r_o^2 - r_i^2}{\ln(r_o/r_i)}$$

Thus, $C_2 = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right) r_o^2 - C_1 \ln r_o$ (From (b))

* $v_z = \frac{1}{4\mu}\left(\frac{\partial P}{\partial z}\right) r^2 + C_1 \ln r + \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right) r_o^2 - C_1 \ln r_o$

$$\Rightarrow v_z = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[r_o^2 - r^2\right] + C_1 \ln\left(r/r_o\right)$$

$$\Rightarrow v_z = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)(r_o^2 - r^2) + \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\frac{r_o^2 - r_i^2}{\ln(r_o/r_i)} \ln(r/r_o)$$

$$\Rightarrow v_z = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[r_o^2 - r^2 + \frac{r_o^2 - r_i^2}{\ln(r_o/r_i)} \ln(r/r_o)\right]$$

* Maximum velocity location?

$$\frac{\partial v_z}{\partial r} = \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[0 - 2r_m + \frac{r_o^2 - r_i^2}{\ln(r_o/r_i)}\left(\frac{r_o}{r_m}\right)\frac{1}{r_o}\right] = 0$$

$$\Rightarrow -2r_m + \frac{r_o^2 - r_i^2}{\ln(r_o/r_i)} \frac{1}{r_m} = 0$$

$$\Rightarrow 2r_m = \frac{r_o^2 - r_i^2}{\ln(r_o/r_i)}\left(\frac{1}{r_m}\right)$$

$$\Rightarrow r_m = \sqrt{\frac{(r_o^2 - r_i^2)}{2\ln(r_o/r_i)}}$$

* Maximum velocity can be obtained by putting $r = r_m$ in v_z expression. (Do it at home !!)

* The flow rate can be obtained as

$$q = \int \vec{v} \cdot d\vec{A} = \int_{r_i}^{r_o} (v_z)\hat{e}_z \cdot (2\pi r\, dr)\hat{e}_z$$

$$\Rightarrow q = \int_{r_i}^{r_o} \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[r_o^2 - r^2 + \frac{r_o^2 - r_i^2}{\ln(r_o/r_i)} \ln(r/r_o)\right] 2\pi r\, dr$$

$$\Rightarrow q = \frac{\pi}{8\mu}\left(-\frac{\partial P}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

Do it @ home

$$\Rightarrow \underbrace{q}_{\text{effect}} = \underbrace{\frac{\pi}{8\mu}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]}_{\text{coefficient relating cause \& effect}} \underbrace{\left(-\frac{\partial P}{\partial z}\right)}_{\text{cause}}$$

* For all pressure driven flow (constan pressure gradient along flow axis) can be shown to take the form $q = G(\Delta P)$

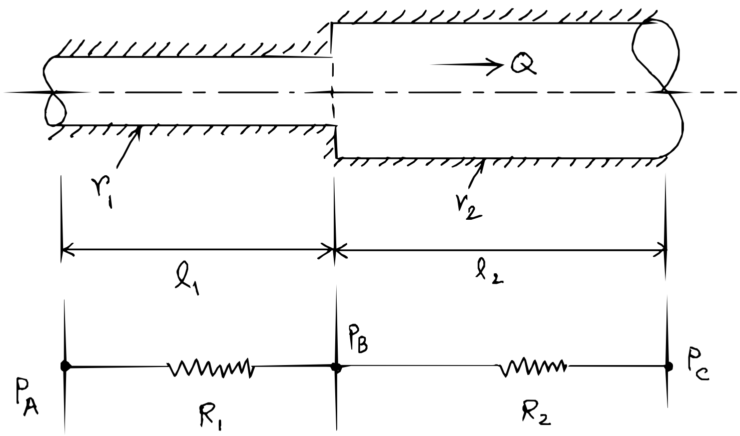
G → Flow condutance (Inverse of resistance)

$$\boxed{\frac{\partial P}{\partial z} = \left(\frac{\Delta P}{l}\right)}$$

Flow	Expression for conductance $\left(G = \frac{1}{R}\right)$
Poiseulli flow (Parallel plate)	$G = \frac{2h^3}{3\mu l}$ (h = plate half gap)
Poisseulli flow (Circular tube)	$G = \frac{\pi R^4}{8\mu l}$ (R = tube radius)
Annular flow (concentric tube)	$G = \frac{\pi}{8\mu l}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$ (r_o & r_i are outer and inner tube radius)

* Flow through tube network :

* Tubes in series



* Here, $R_1 = \left(\dfrac{8\mu l_1}{\pi r_1^4}\right)$, $R_2 = \left(\dfrac{8\mu l_2}{\pi r_2^4}\right)$

Thus, $Q = \left(\dfrac{P_A - P_B}{R_1}\right) = \left(\dfrac{P_B - P_C}{R_2}\right)$

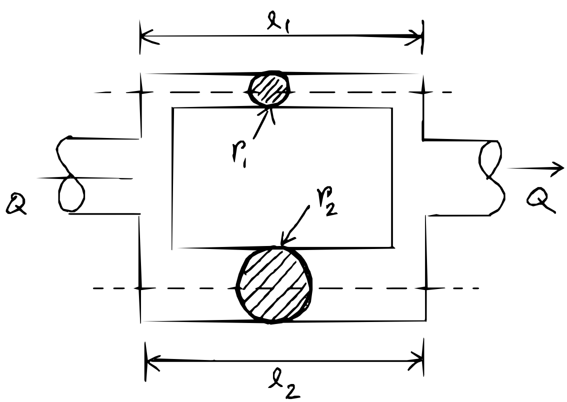
$$\Rightarrow P_A - P_B = Q R_1 , \quad P_B - P_C = Q R_2$$

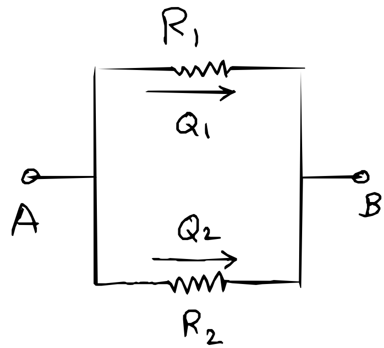
$$(+) \; P_A - P_C = Q(R_1 + R_2)$$

$$\Rightarrow Q = \left(\dfrac{P_A - P_C}{R_1 + R_2}\right) = \left(\dfrac{1}{R_{eq}}\right)(P_A - P_C)$$

Then, $R_{eq} = R_1 + R_2$ (series)

* Tubes in parallel:

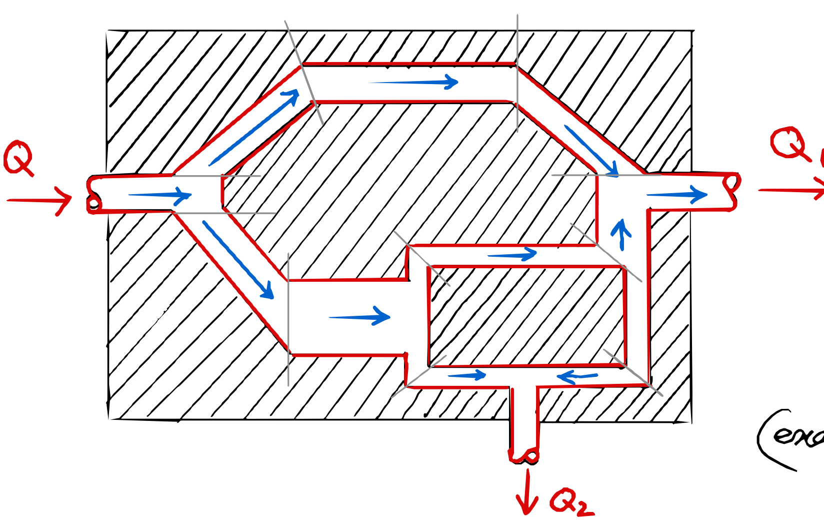




* $Q = Q_1 + Q_2$

$$\rightarrow \left(\frac{P_A - P_B}{R_{eq}}\right) = \left(\frac{P_A - P_B}{R_1}\right) + \left(\frac{P_A - P_B}{R_2}\right) \Rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \text{ (parallel)}$$

* Network of tubes:



(example)