Ch-5

Conservation of mass (Continuity equation)

* Remmember Reynold's transport theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \, \vec{v} \cdot d\vec{A}$$

For mass, $B_{sys} = M_{sys}$ (total mass in the system).

& $b = 1 \longrightarrow$ (mass specific mass)

Thus, $M_{sys} = \int_{sys} \rho \, d\forall$

* Substitution,

$$\frac{D}{Dt}\int_{sys} \rho \, d\forall = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{v} \cdot d\vec{A}$$

* Conservation of mass demands that

$$\frac{D}{Dt}\int_{sys} \rho \, d\forall = 0$$

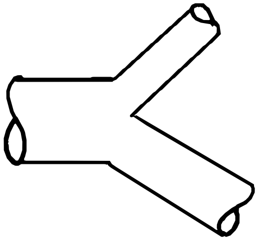
$$\Rightarrow \boxed{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0}$$

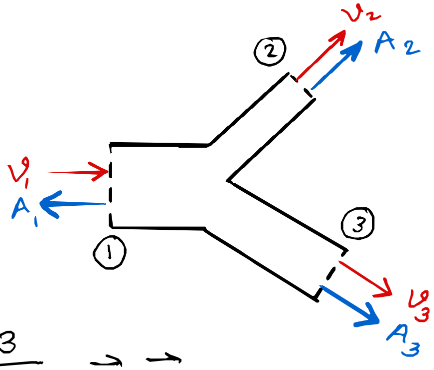
→ Is it a vector/ scalar equation ?

* Special cases:

(a) Steady flow $\longrightarrow \frac{\partial}{\partial t} = 0$

$$\therefore \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0$$





$$* \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0 \Rightarrow \sum_{i=1}^{3} \rho_i \vec{v}_i \cdot \vec{A}_i = 0$$

$$\Rightarrow -\rho_1 v_1 A_1 + \rho_2 v_2 A_2 + \rho_3 v_3 A_3 = 0$$

$$\Rightarrow \rho_1 v_1 A_1 = \rho_2 v_2 A_2 + \rho_3 v_3 A_3$$

(b) steady incompressible flow $\longrightarrow \frac{\partial}{\partial t} = 0, \rho = \text{const.}$

$$\int \vec{v} \cdot d\vec{A} = 0 \xrightarrow[\text{case}]{\text{above}} v_1 A_1 = v_2 A_2 + v_3 A_3$$

(c) Unsteady incompressible flow $\longrightarrow \frac{\partial}{\partial t} \neq 0, \rho = \text{const.}$

$$\frac{\partial}{\partial t} \int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(\forall)}_{\text{rate of volume change}} + \underbrace{\int \vec{v} \cdot d\vec{A}}_{} = 0$$

$\longrightarrow$ volume flow rate out − volume flow rate in.

Differential form of continuity equation

* We have,

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0$$

* Divergence theorem: $\int_{cv} \nabla \cdot \vec{\phi} \, d\forall = \int_{cs} \vec{\phi} \cdot d\vec{A}$

Thus, $$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cv} \nabla \cdot (\rho \vec{v}) \, d\forall = 0$$

$$\Rightarrow \int_{cv} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \, d\forall = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$ (Scalar equation)

* We will prove this again using differential fluid volume approach.

Things to remmember

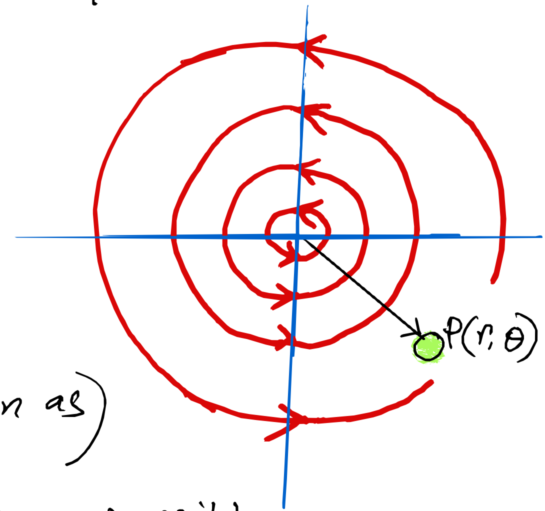
* The mass balance: $\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0$

or, $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

* For steady incompressible flow;

$$\nabla \cdot \vec{v} = 0$$

* The velocity field of a pure vortex is given as:



$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$$

where, $v_r = 0$

and $v_\theta = \left(\Gamma/2\pi r\right)$

(Γ = a constant known as vortex strength)

Check that the flow is incompressible

Sol^n: For steady incompressible flow $\nabla\cdot\vec{u} = 0$

$$\Rightarrow \left[\left(\frac{\partial}{\partial r}\right)\hat{e}_r + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\hat{e}_\theta\right]\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}(v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(v_\theta) = 0$$

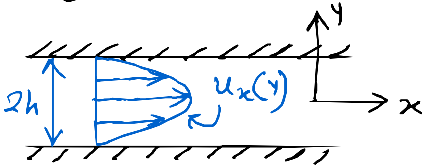
$$\Rightarrow \quad 0 = 0 \quad \text{(checked)}$$

* The velocity field for flow between two parallel plates due to constant pressure gradient (Poiseuille flow) is known as

$$\vec{u} = u_x\hat{i} + u_y\hat{j} \quad (u_y = 0)$$

where $u_x = \frac{3}{2}U\left[1 - \left(\frac{y}{h}\right)^2\right]$

(U is the average velocity which is constant)



* show that $\nabla\cdot\vec{u} = 0$

$$\vec{u} = \frac{3U}{2}\left[1 - \frac{y^2}{h^2}\right]\hat{i}$$

Thus, $\nabla \cdot \vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot \left[\left(\frac{3U}{2} - \frac{3U}{2}\frac{y^2}{h^2}\right)\hat{i} + 0\hat{j}\right]$

$$= \frac{\partial}{\partial x}\left(\frac{3U}{2} - \frac{3U}{2}\frac{y^2}{h^2}\right)$$

$$= 0$$

* Try to reconsile in your head what will happen when the flow is compressible $\nabla \cdot \vec{u} \neq 0$.

Hint, use mass balance equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

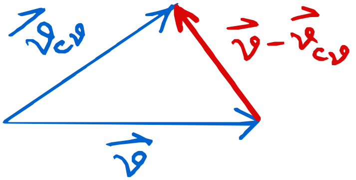
→ From here define compressible flow

→ why the equation above do not have volume (rather includes density) while compressibility is the sensitivity of volume change for pressure change? (See lecture for Chapter 1)

* Moving Control Volume and mass balance

→ Mass balance in a moving control volume

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \underbrace{(\vec{v}-\vec{v}_{cv})}_{\text{relative velocity vector}} \cdot d\vec{A} = 0$$

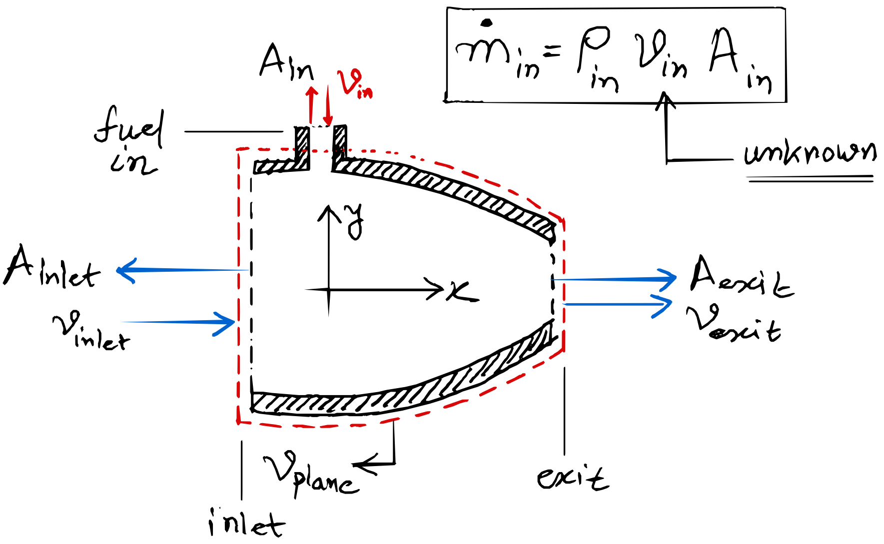


→ only velocity vector that matters !!

* steady incompressible flow in a moving cv

$$\int_{cs} (\vec{v}-\vec{v}_{cv}) \cdot d\vec{A} = 0$$

* Example 5.6 (Textbook)



$$\boxed{\dot{m}_{in} = \rho_{in} v_{in} A_{in}}$$

* Moving cv approach

$\vec{v}_{cv} = -v_{plane}\,\hat{i}$

$\vec{v}_{exit} = v_{exit}\,\hat{i}$

$\vec{v}_{inlet} = 0$

$\vec{v}_{in} = -v_{in}\,\hat{j}$

$\vec{A}_{exit} = A_{exit}\,\hat{i}$

$\vec{A}_{inlet} = -A_{inlet}\,\hat{i}$

$\vec{A}_{in} = +A_{in}\,\hat{j}$

Thus, $\vec{v}_{exit} - \vec{v}_{cv} = (v_{exit} + v_{plane})\,\hat{i}$

$\vec{v}_{inlet} - \vec{v}_{cv} = +v_{plane}\,\hat{i}$

$\vec{v}_{in} - \vec{v}_{cv} = -v_{in}\,\hat{j} + v_{plane}\,\hat{i}$

Again, $(\vec{v}_{exit} - \vec{v}_{cv})\cdot\vec{A}_{exit} = (v_{exit} + v_{plane})\,A_{exit}$

$(\vec{v}_{inlet} - \vec{v}_{cv})\cdot\vec{A}_{inlet} = -(v_{plane})\,A_{inlet}$

$(\vec{v}_{in} - \vec{v}_{cv})\cdot\vec{A}_{in} = -(v_{in})\,A_{in}$

See automatically isolates inflow and outflow as $-$(ve) and $+$(ve).

From mass balance:

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{cv}\rho\, d\forall}}_{0\ (\text{steady})} + \int_{cs}\rho(\vec{v} - \vec{v}_{cv})\cdot d\vec{A} = 0$$

the plane is moving at constant velocity. Nothing inside cv changes

* Since we only have 3 control surfaces that mass are allowed to cross, the mass balance can written as

$$\sum_{i=\text{inlet, exit, feed in}} \rho_i \left(\vec{v}-\vec{v}_{cv}\right)_i \cdot \vec{A}_i = 0$$

$$\Rightarrow \rho_{exit}\left(V_{exit}+V_{plane}\right)A_{exit} - \rho_{inlet}\left(v_{plane}\right)A_{inlet} - \rho_{in}\, v_{in}\, A_{in} = 0$$

$$\Rightarrow \dot{m}_{in} = \rho_{exit}\left(V_{exit}+V_{plane}\right)A_{exit} - \rho_{inlet}\left(V_{plane}\right)A_{inlet}$$

$$\Rightarrow \dot{m}_{in} = \rho_{inlet}\, V_{plane}\, A_{inlet}\left[\frac{\rho_{exit}}{\rho_{inlet}}\left(\frac{V_{exit}}{V_{plane}}+1\right)\frac{A_{exit}}{A_{inlet}} - 1\right]$$

* Substitution of values gives

$\rho_{inlet} = 0.736\ kg/m^3$	$\rho_{exit} = 0.515\ kg/m^3$
$V_{plane} = 971\ km/hr$	$V_{exit} = 1050\ km/hr$
$\approx 269.72\ m/s$	$\approx 291.6\ m/s$
$A_{inlet} = 0.8\ m^2$	$A_{exit} = 0.558\ m^2$

$$\dot{m}_{in} \approx 2.51\ kg/s \approx 9049.9\ kg/hr$$ (Ans)

Conservation of linear momentum

* From momentum balance we know

$$\sum \vec{F}_{sys} = \frac{D}{Dt}\int_{sys} \rho \vec{v}\, d\forall \quad \left(\sum \vec{F} = m\vec{a}\right)$$

$\hookrightarrow$ Force causing change

$$\Rightarrow \frac{D}{Dt}\int_{sys} \rho \vec{v}\, d\forall = \,?$$

From particle perspective to continuum.

* From Raynolds transport theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b\, d\forall + \int_{cs} \rho b\, \vec{v}\cdot d\vec{A}$$

Setting $\vec{B}_{sys} = m\vec{v}$ and $b = \vec{v}$ (mass sp. momentum)

$$\vec{B}_{sys} = \int_{sys} \rho \vec{v}\, d\forall \quad \left(\text{volume integral over system}\right)$$

Thus,

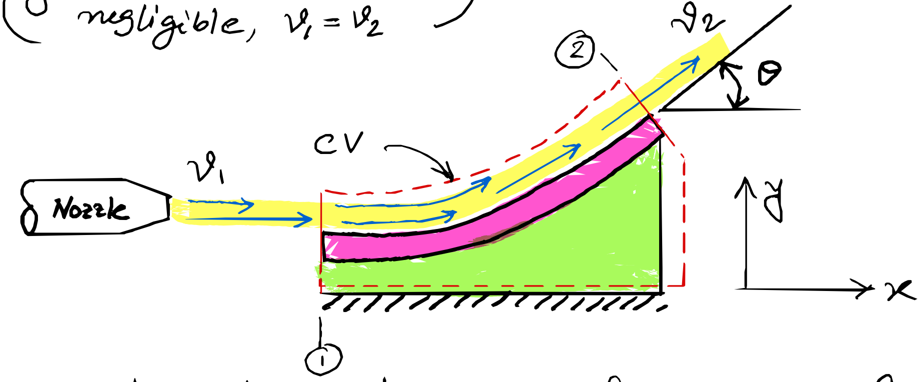
$$\underbrace{\frac{D}{Dt}\int_{sys} \rho \vec{v}\, d\forall}_{\boxed{\text{rate of change of momentum in the system}}} = \underbrace{\frac{\partial}{\partial t}\int_{cv} \rho \vec{v}\, d\forall}_{\boxed{\text{rate of change of momentum in the cv}}} + \underbrace{\int_{cs} (\rho \vec{v})\, \vec{v}\cdot d\vec{A}}_{\boxed{\text{net outflow rate of momentum through cs}}}$$

$$\Rightarrow \sum F_{sys} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{v}\, d\forall + \int_{cs} \vec{v}\rho\, \vec{v}\cdot dA \quad \left(\text{Vector equation}\right)$$

$\hookrightarrow$ Force that is required for accelerction (maintaining the flow)

* Consider Jet hitting a deflector. (Example 5.10)

(g and viscous effects negligible, $v_1 = v_2$)



* we want to know how much force we need to hold the deflector in place.

* Is $A_1 = A_2$? conservation of mass

$$\frac{\partial}{\partial t}\cancelto{0}{\int_{cv} \rho \, d\forall} + \int_{cs} \rho \vec{v} \cdot d\vec{A} = 0$$

$$\Rightarrow \quad \rho_2 v_2 A_2 - \rho_1 v_1 A_1 = 0 \quad \Rightarrow A_1 = A_2 \quad (\rho = const)$$

* From force/momentum balance we know

$$\sum F_{sys} = \frac{\partial}{\partial t}\cancelto{0}{\int_{cv} \rho \vec{v} \, d\forall} + \int_{cs} \vec{v} \rho \vec{v} \cdot d\vec{A}$$

$$\Rightarrow \sum F_{sys} = \int_{cs} \vec{v} \rho \vec{v} \cdot d\vec{A} = \sum_{cs} \vec{v}_i \rho (\vec{v}_i \cdot \vec{A}_i)$$

at cs-1: $A_1 = -A_1 \hat{i}$, $v_1 = v_1 \hat{i}$

at cs-2: $A_2 = A_1 \cos\theta \, \hat{i} + A_1 \sin\theta \, \hat{j}$

$$v_2 = v_1 \cos\theta \, \hat{i} + v_1 \sin\theta \, \hat{j}$$

Thus, $\sum \vec{F}_{sys} = v_1 \hat{i} \cdot \rho \left(-v_1 A_1\right) + \left(v_1 \cos\theta \, \hat{i} + v_1 \sin\theta \, \hat{j}\right)$

$$\rho\left(v_1 A_1 \cos^2\theta + v_1 A_1 \sin^2\theta\right)$$

$$\Rightarrow \sum \vec{F}_{sys} = -\left(\rho A_1 v_1^2\right)\hat{i} + \rho v_1 A_1 \left(v_1 \cos\theta \, \hat{i} + v_1 \sin\theta \, \hat{j}\right)$$

$$\Rightarrow \sum \vec{F}_{sys} = -\left(\rho A_1 v_1^2\right)\hat{i} + \left(\rho A_1 v_1^2 \cos\theta\right)\hat{i} + \left(\rho A_1 v_1^2 \sin^2\theta\right)\hat{j}$$

$$\Rightarrow \sum \vec{F}_{sys} = \rho A_1 v_1^2 \left(\cos\theta - 1\right)\hat{i} + \rho A_1 v_1^2 \left(\sin\theta\right)\hat{j}$$

Thus, the force in x-axis $\sum F_x = \rho A_1 v_1^2 \left(\cos\theta - 1\right)$

the force in y-axis $\sum F_y = \rho A_1 v_1^2 \left(\sin\theta\right)$

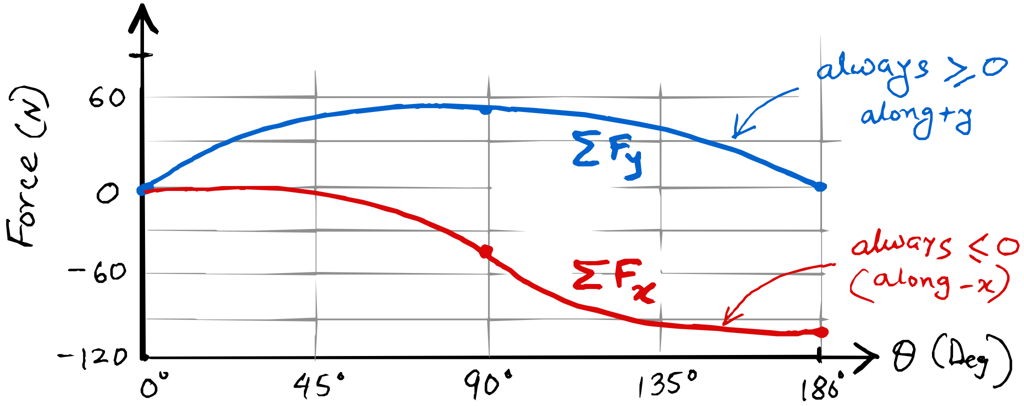
Putting $A_1 = 0.06 \text{ ft}^2 = 0.00557 \text{ m}^2$

$v_1 = 10 \text{ ft/s} = 3.048 \text{ m/s}$

$$\sum F_x = 10^3 \times 0.00557 \times (3.048)^2 \left(\cos\theta - 1\right) = 51.75\left(\cos\theta - 1\right)$$

$$\sum F_y = 10^3 \times 0.00557 \times (3.048)^2 \left(\sin\theta\right) = 51.75\left(\sin\theta\right)$$

* These F_x & F_y are the forces that is required to maintain the flow to curve/bend at angle θ



* what if the cart is moving at a constant velocity v_{cart} to the right.

→ Does the forcees due to the jet changes?

* The control volume now moves.

$$\rightarrow \sum \vec{F}_{sys} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{v}\, d\forall + \int_{cs} \vec{v}\rho\left(\vec{v}-\vec{v}_{cv}\right)\cdot d\vec{A}$$

Explain why (Important)	ⓐ these do not get replaced by relative velocity?	ⓑ changes to relative velocity!!

* Now if the flow is steady,

$$\sum \vec{F}_{sys} = \int_{cs} \vec{v}\rho\left(\vec{v}-\vec{v}_{cv}\right)\cdot d\vec{A}$$

Here, $\vec{v}_1 = v_1\hat{i}$, $\vec{A}_1 = -A_1\hat{i}$

$v_2 = v_1\cos\theta\,\hat{i} + v_1\sin\theta\,\hat{j}$

$\vec{A}_2 = A_1\cos\theta\,\hat{i} + A_1\sin\theta\,\hat{j}$

$\vec{v}_{cart} = v_{cart}\,\hat{i}$

For cs-1: $\vec{v}_1\rho\left(\vec{v}_1-\vec{v}_{cv}\right)\cdot\vec{A}_1 = -v_1\rho\left(v_1-v_{cart}\right)A_1\hat{i}$

$$= -\left(\rho A_1 v_1^2 - \rho A_1 v_1 v_{cart}\right)\hat{i}$$

cs-2: $\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)\rho\left[\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)-\left(v_{cart}\,\hat{i}\right)\right]\cdot\left(A_1\cos\theta\,\hat{i}+A_1\sin\theta\,\hat{j}\right)$

$$cs-2: \left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)\rho\left[\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right) - \left(v_{cart}\,\hat{i}\right)\right]\cdot\left(A_1\cos\theta\,\hat{i}+A_1\sin\theta\,\hat{j}\right)$$

$$= \rho\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)\left[\left(v_1\cos\theta - v_{cart}\right)\hat{i}+\left(v_1\sin\theta\right)\hat{j}\right]\cdot\left(A_1\cos\theta\,\hat{i}+A_1\sin\theta\,\hat{j}\right)$$

$$= \rho\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)\left(A_1 v_1\cos^2\theta - A_1 v_{cart}\cos\theta + A_1 v_1\sin^2\theta\right)$$

$$= \rho\left(A_1 v_1 - A_1 v_{cart}\cos\theta\right)\left(v_1\cos\theta\,\hat{i}+v_1\sin\theta\,\hat{j}\right)$$

$$= \left(\rho A_1 v_1^2\cos\theta - \rho A_1 v_1 v_{cart}\cos\theta\right)\hat{i}+\left(\rho A_1 v_1^2\sin\theta - \rho A_1 v_1 v_{cart}\sin\theta\right)\hat{j}$$

Thus,

$$\sum\vec{F}_{sys} = \left[\left(\rho A_1 v_1^2\cos\theta - \rho A_1 v_1 v_{cart}\cos\theta\right) - \left(\rho A_1 v_1^2 - \rho A_1 v_1 v_{cart}\right)\right]\hat{i} + \left(\rho A_1 v_1^2\sin\theta - \rho A_1 v_1 v_{cart}\sin\theta\right)\hat{j}$$

$$\Rightarrow \sum F_x = \rho A_1 v_1\left[v_1\cos\theta - v_{cart}\cos\theta - v_1 + v_{cart}\right]$$

$$= \rho A_1 v_1^2(\cos\theta - 1) - \rho A_1 v_1 v_{cart}(\cos\theta - 1)$$

$$= \rho A_1 v_1^2(\cos\theta - 1)\left[1 - \left(\frac{v_{cart}}{v_1}\right)\right]$$

$$\& \sum F_y = \rho A_1 v_1^2(\sin\theta)\left[1 - \left(\frac{v_{cart}}{v_1}\right)\right]$$

$$* \sum F_x = \rho A_1 v_1^2 (\cos\theta - 1)\left[1 - \left(\frac{v_{cart}}{v_1}\right)\right]$$

$$\sum F_y = \rho A_1 v_1^2 (\sin\theta)\left[1 - \left(\frac{v_{cart}}{v_1}\right)\right]$$

* Does these equation make sense ?

ⓐ when, $v_{cart} = 0$ (stationery cart)

$$\left.\begin{aligned} \sum F_x &= \rho A_1 v_1^2 (\cos\theta - 1) \\ \sum F_y &= \rho A_1 v_1^2 (\sin\theta) \end{aligned}\right\}$$

same as stationary cart example

ⓑ when, $v_{cart} = v_1$

$$\left.\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \end{aligned}\right\}$$

No force due to water jet ??

ⓒ when, $\theta = 0°$

$$\left.\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \end{aligned}\right\}$$

No force at all

ⓓ when $\theta = 90°$

$$\sum F_x < 0$$

$$\sum F_y > 0$$

Conservation of Energy

* The first law of thermodynamics states,

$$\boxed{\begin{array}{c}\text{Time rate of}\\ \text{change of the}\\ \text{total energy in}\\ \text{the system}\end{array}} = \boxed{\begin{array}{c}\text{Net time rate}\\ \text{of energy addition}\\ \text{into the system}\end{array}} + \boxed{\begin{array}{c}\text{Net time rate}\\ \text{of work done}\\ \text{to the system}\end{array}}$$

$$\Rightarrow \quad \frac{D}{Dt}\left(E_{sys}\right) = \sum \dot{Q}_{in} + \sum \dot{W}_{in}$$

* From Raynolds transport theorem

$$\frac{D}{Dt}\left(B_{sys}\right) = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \, \vec{v}\cdot d\vec{A}$$

* For $B = E$ and $b = \left(E/m\right) = e \leftarrow \left(\begin{array}{c}\text{mass sp.}\\ \text{energy}\end{array}\right)$

$$\frac{D}{Dt}\left(E_{sys}\right) = \frac{\partial}{\partial t}\int_{cv} \rho e \, d\forall + \int_{cs} \rho e \, \vec{v}\cdot d\vec{A}$$

$$\Rightarrow \quad \frac{D}{Dt}\int_{sys} \rho e \, d\forall = \frac{\partial}{\partial t}\int_{cv} \rho e \, d\forall + \int_{cs} \rho e \, \vec{v}\cdot d\vec{A}$$

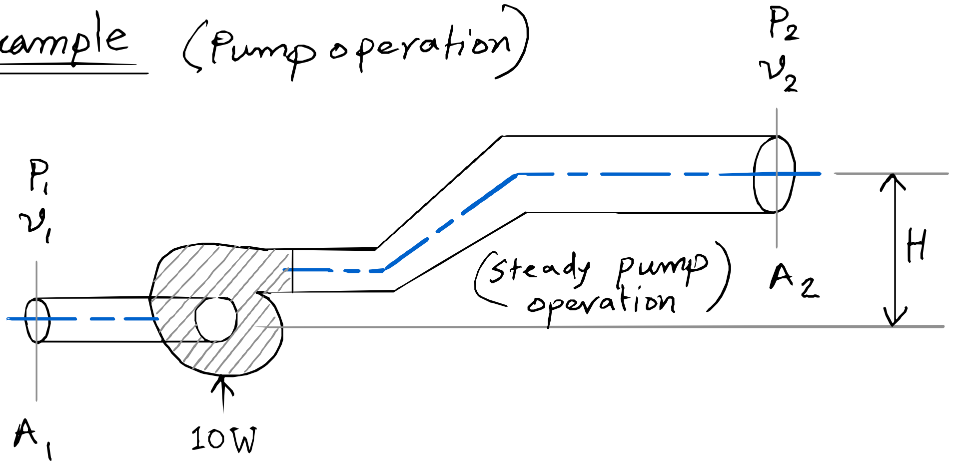
*

$$\boxed{\frac{\partial}{\partial t}\int_{cv} \rho e \, d\forall + \int_{cs} \rho e \, \vec{v}\cdot d\vec{A} = \sum \dot{Q}_{in} + \sum \dot{W}_{in}}$$

* Remmember e is mass specific energy with unit (J/kg)

Example (Pump operation)

ⓐ



* mass specific total energy at the inlet and outlet

$$e_1 = \left(\frac{P_1}{\rho}\right) + \left(\frac{v_1^2}{2}\right) + (g z_1)$$

$$e_2 = \left(\frac{P_2}{\rho}\right) + \left(\frac{v_2^2}{2}\right) + g z_2$$

* Thus,

$$\cancelto{0}{\frac{\partial}{\partial t}\int_{cv} \rho e \, d\forall} + \int_{cs} \rho e \, \vec{v} \cdot d\vec{A} = 10$$

$$\Rightarrow -\left(P_1 + \frac{1}{2}\rho v_1^2\right) \cdot v_1 A_1 + \left(P_2 + \frac{1}{2}\rho v_2^2 + \rho g H\right) v_2 A_2 = 10$$

$$\Rightarrow H = \frac{1}{\rho g}\left[\left(\frac{10 + P_1 v_1 A_1 + \frac{1}{2}\rho v_1^3 A}{v_2 A_2}\right) - \left(P_2 + \frac{1}{2}\rho v_2^2\right)\right]$$