Ch-4

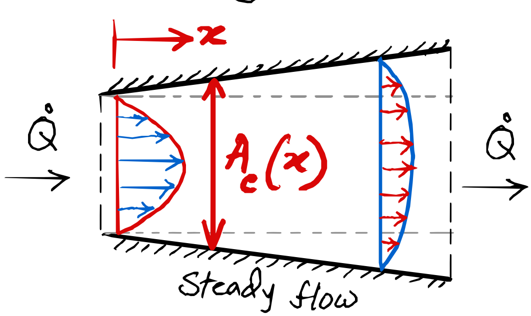
Eulerian Vs Lagrangian Description

Fluid Mechanics Problems

- Eulerian
- Lagrangian

Eulerian	Lagrangian
⊛ Field concept	⊛ Particle concept
→ Properties are defined over certain volumes	→ Properties are defined for particles

* How would you get the velocity field for the following 2-D steady flow (you have following technology available)



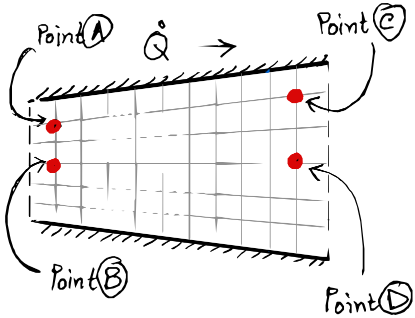
$$\dot{Q} = \int \vec{v} \cdot d\vec{A}$$

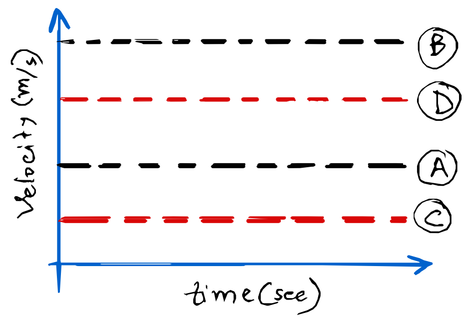
Simple form $\dot{Q} = Av$

⊛ You have unlimited number of sensors/probes capable of measuring velocity of individual molecules.

⊛ You have unlimited time to conduct your experiment.

* **Approach-1:** Place the probes at different locations and measure the velocity at those locations as a function of time.



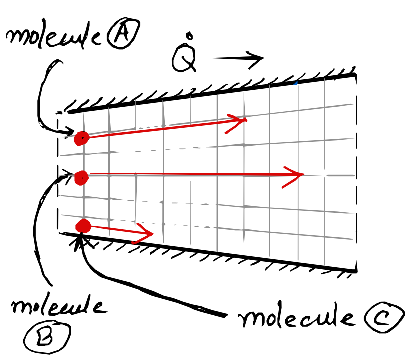


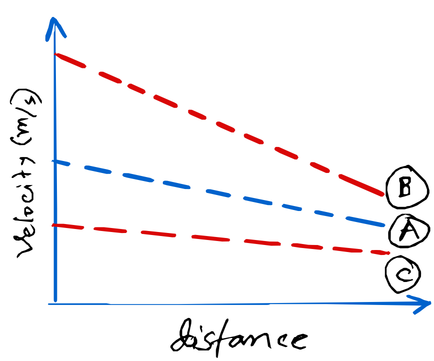
(Eulerian)

* How to obtain average velocity from this data?

⟶ Area average velocity $\vec{v} = \frac{1}{A_c} \int \vec{v} \cdot d\vec{A}$

* **Approach-2:** Attach the proves to different molecules and measure the velocity of those molecules as function of space.





(Lagrangian)

* Area average velocity (magnitude only) $v = \frac{1}{t_{exp}} \int v \, dt$

(*) Eulerian approach is easy and less time consuming. (Both experimentally & theoritically)
(Thus widely used in fluid mechanics)

Convective acceleration

(*) The total acceleration is,

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \text{(defined in Lagrangian concept)}$$

we know that $\vec{v}$ can be a function of space (x, y, z) and time simulteneously.

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + \left(\frac{\partial \vec{v}}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial \vec{v}}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial \vec{v}}{\partial z}\right)\frac{dz}{dt}$$

$$\Rightarrow \vec{a} = \frac{\partial \vec{v}}{\partial t} + v_x \frac{\partial \vec{v}}{\partial x} + v_y \frac{\partial \vec{v}}{\partial y} + v_z \frac{\partial \vec{v}}{\partial z}$$

$$\Rightarrow \vec{a} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

Remmember: $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

* Vector identity $\vec{a} \cdot \nabla \vec{b} = (\vec{a} \cdot \nabla)\vec{b}$

$$\therefore \vec{a} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{(transient acceleration)}} + \underbrace{\vec{v} \cdot \nabla \vec{v}}_{\text{(Convective acceleration)}}$$

* what does the convective acceleration mean?

→ Check the unit of $(\vec{v}\cdot\nabla)\vec{v}$ or, $(\vec{v}\cdot\nabla\vec{v})$

→ why using $(\vec{v}\cdot\nabla)\vec{v}$ is more convenient than using $\vec{v}\cdot(\nabla\vec{v})$.

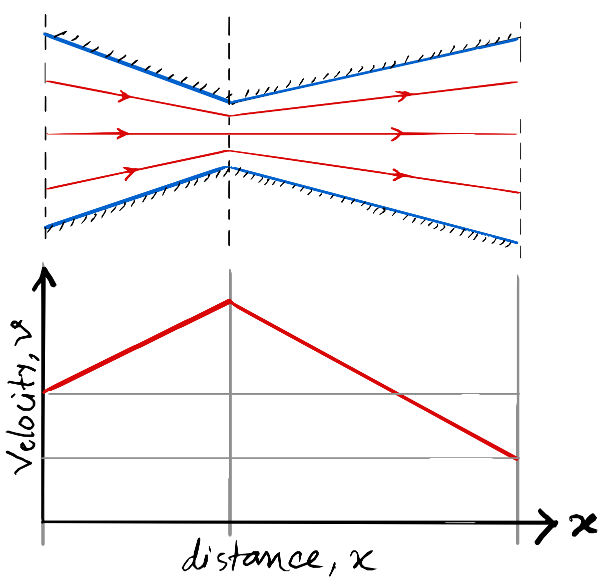
* Demonstration: if $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$

then, determine
ⓐ $(\vec{v}\cdot\nabla)\vec{v}$
ⓑ $\vec{v}\cdot(\nabla\vec{v})$

→ what is the problem you are facing while dealing with ⓑ?

* Where is it coming from?

→ Consider a steady flow through a C-D nozzle.



ⓐ steady flow:

(Nothing changes with time) !!

ⓑ why/how velocity increases/decreases in the converging/diverging sections respectively?

* Later on we will see the operator below

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right) = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

appears in many other fundamental eqns.

* Thus, we introduce a new type of derivative operator "Material derivative" (also called "Substantial derivative")

$$\frac{D}{Dt} \approx \left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)$$

convective derivative

time derivative

material derivative

* Stream line co-ordinate

→ Revisit ch-3 lecture where we showed that

$a_s = v\dfrac{\partial v}{\partial s}$ & assumed $a_n = \dfrac{v^2}{R_c}$.

* Remember in streamline co-ordinate

$\vec{v} = v\hat{s}$ | here we use $\hat{s}$ and $\hat{n}$ as unit vectors

* Now, $\vec{a} = \dfrac{D\vec{v}}{Dt} = \dfrac{D(v\hat{s})}{Dt} = \left(\dfrac{Dv}{Dt}\right)\hat{s} + v\left(\dfrac{D\hat{s}}{Dt}\right)$

$$\Rightarrow \vec{a} = \left(\frac{Dv}{Dt}\right)\hat{s} + v\left(\frac{D\hat{s}}{Dt}\right)$$

$$\Rightarrow \vec{a} = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}\right)\hat{s} + v\left(\frac{\partial \hat{s}}{\partial t} + v\frac{\partial \hat{s}}{\partial s}\right)$$

For steady flow $\longrightarrow \dfrac{\partial v}{\partial t} = 0$, and $\dfrac{\partial \hat{s}}{\partial t} = 0$

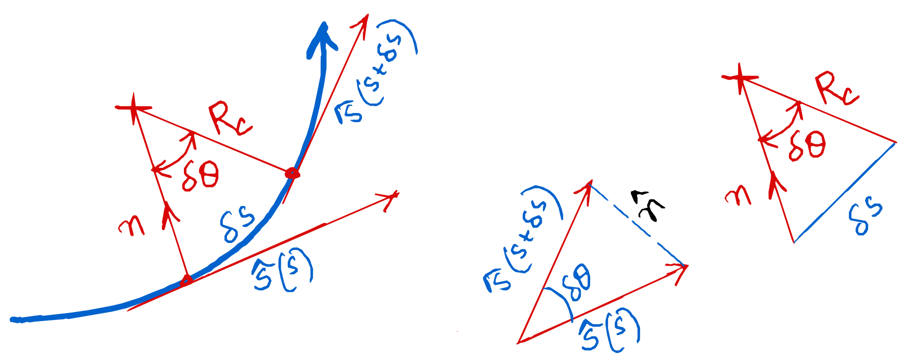
Thus, $\vec{a} = \left(v\dfrac{\partial v}{\partial s}\right)\hat{s} + v\left(v\dfrac{\partial \hat{s}}{\partial s}\right)$

what is $\underline{\dfrac{\partial \hat{s}}{\partial s}}$? (does this term look odd)

It represents change in unit vector direction as we move along streamline

Thus for a straight streamline $\dfrac{\partial \hat{s}}{\partial s} = 0$.

* what happens for curved streamline



* Since the two triangle's are similar,

$$\left(\frac{\delta s}{R_c}\right) = \left(\frac{\delta \hat{s}}{\hat{n}}\right) \Rightarrow \frac{\delta \hat{s}}{\delta s} = \frac{\hat{n}}{R_c}$$

From substitution we get,

$$\vec{a} = \left(v \frac{\partial v}{\partial s}\right) \hat{s} + \left(v^2/R_c\right) \hat{n}$$

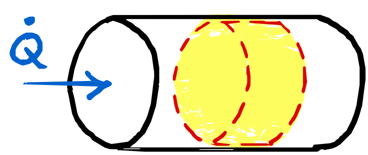
Control Volume representation

* System and control volume

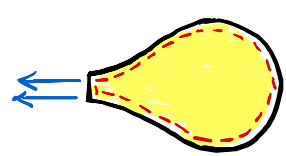
System → Collection of matter of fixed identity (same atoms or particles always)

Control volume → Volume in space (pure geometric entity) atoms or particle can pass/flow through it (mass independent)

* The boundaries of a control volume is called control surface.
* A control volume can be stationery/moving also it can change shape/size.



Stationary CV



Deforming CV

Raynold's Transport Theorem

* It is a tool to switch from system representation to control volume representation of a physical process.

* Any physical property can be expressed in two forms (a) Extensive property
  (b) Intensive property (mass specific)

* Thus, $\begin{pmatrix}\text{Extensive}\\ \text{property}\end{pmatrix} = (\text{mass})\begin{pmatrix}\text{intensive}\\ \text{property}\end{pmatrix}$

$$\Rightarrow \quad B = m\, b \quad (\text{General form})$$

* For a system, $B_{sys} = \int_{sys} \rho b \, d\forall$

$$\Rightarrow \left(\frac{dB_{sys}}{dt}\right) = \left(\frac{d \int_{sys} \rho b \, d\forall}{dt}\right)$$

* This equation represent rate of change in the system.

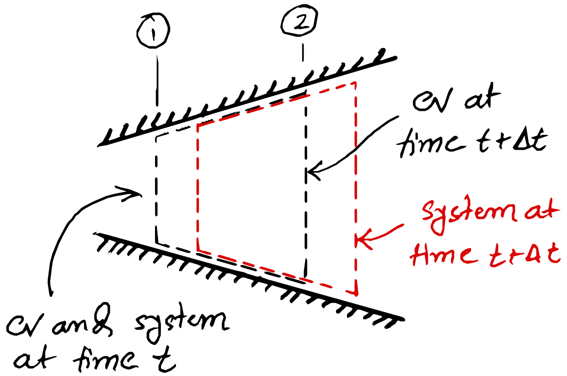
* For a control volume, $B_{cv} = \int_{cv} \rho b \, d\forall$

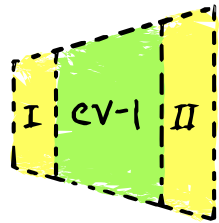
$$\Rightarrow \left(\frac{d B_{cv}}{dt}\right) = \left(\frac{d \int_{cv} \rho b \, d\forall}{dt}\right)$$

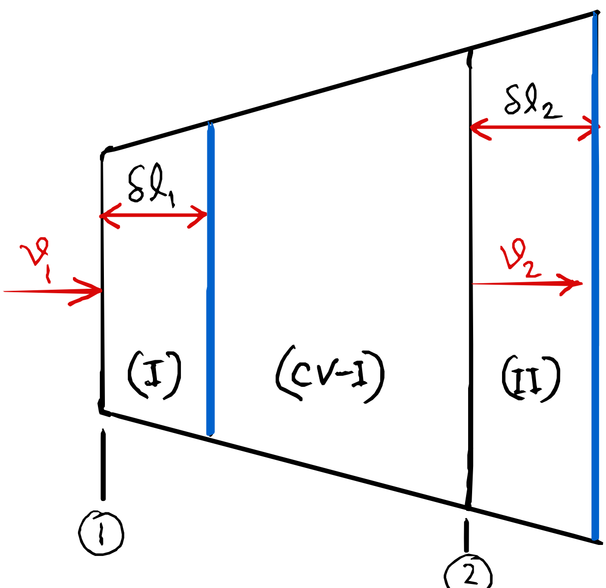
* This equation represent rate of change in control volume.

Relation between $\left(\frac{dB_{sys}}{dt}\right)$ & $\left(\frac{dB_{cv}}{dt}\right)$

* Consider a flow through diverging tube.







$$\delta l_1 = v_1 \delta t$$

$$\delta l_2 = v_2 \delta t$$

* At time 't' the system and the control volume contains same fluid particles. Thus

$$B_{sys}(t) = B_{cv}(t)$$

* The system changes during time $t \sim t+\delta t$. This can be obtained as,

$$B_{sys}(t+\delta t) = B_{cv}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t)$$

$$\Rightarrow \left(\frac{\delta B_{sys}}{\delta t}\right) = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t}$$

$$\Rightarrow \left(\frac{\delta B_{sys}}{\delta t}\right) = \left\{\frac{B_{cv}(t+\delta t) - B_{I}(t+\delta t) + B_{II}(t+\delta t) - B_{cv}(t)}{\delta t}\right\}$$

$$\Rightarrow \left(\frac{\delta B_{sys}}{\delta t}\right) = \left(\frac{\delta B_{cv}}{\delta t}\right) - \left\{\frac{B_{I}(t+\delta t)}{\delta t}\right\} + \left\{\frac{B_{II}(t+\delta t)}{\delta t}\right\}$$

* In the limit $\delta t \to 0$

$$\underbrace{\frac{\delta B_{sys}}{\delta t} = \frac{D B_{sys}}{D t}}_{\text{Lagrangian concept}}, \qquad \underbrace{\frac{\delta B_{cv}}{\delta t} = \frac{\partial B_{cv}}{\partial t}}_{\text{Eulerian concept}}$$

* Again, $B_{II}(t+\delta t) = (\rho_2 b_2)(\delta \forall_{II}) = (\rho_2 b_2) A_2 (\delta l_2)$

$$= \rho_2 b_2 A_2 V_2 \delta t$$

and, $B_{I}(t+\delta t) = \rho_1 b_1 A_1 V_1 \delta t$

* Thus, $$\frac{D B_{sys}}{D t} = \left(\frac{\partial B_{cv}}{\partial t}\right) + \underbrace{(\rho_2 A_2 v_2) b_2}_{B_{out}} - \underbrace{(\rho_1 A_1 v_1) b_1}_{B_{in}}$$

* The term $(\rho_2 A_2 v_2) b_2$ and $(\rho_1 A_1 v_1) b_1$ are respectively the amount of out flow and inflow of intensive property 'b' through the control surfaces.

* Thus Raynolds transport theorem bassically tell us the following.

$$\begin{pmatrix}\text{Rate of change} \\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Rate of change} \\ \text{in control volume}\end{pmatrix} + \begin{pmatrix}\text{Net outflow through} \\ \text{control surfaces}\end{pmatrix}$$

* This expression can be simply put in mathematical notation as

$$\boxed{\left(\frac{DB_{sys}}{Dt}\right) = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \vec{v} \cdot d\vec{A}}$$

→ This expression is very intuitive.

→ It allows us to switch our analysis from particle (molecules) to continuum perspective.

* Now what will happen for a moving cv that is moving at speed v_{cv}?

$$\left(\frac{DB_{sys}}{Dt}\right) = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \underbrace{(\vec{v} - \vec{v}_{cv})}_{\text{Relative velocity}} \cdot d\vec{A}$$

Physical interpretation

* Read section 4.4.2 ~ 4.4.7 from text book.

$\longrightarrow$ we will revesit it's interpretation in chapter 5.

* The extensive quantity can be expressed as a volume integral of the Intensive property as

$$B_{sys} = \int_{sys} b\, dm = \int_{sys} \rho b\, d\forall$$

* $$\frac{D}{Dt}\int_{sys} \rho b\, d\forall = \frac{\partial}{\partial t}\int_{cv} \rho b\, d\forall + \int_{cs} \rho b\, \vec{v}\cdot dA$$

Rate of change in system = Rate of change in control volume + Rate of inflow through the control surface