Ch-2

"Big Idea"

* The "Big Idea" of fluid mechanics is to apply $\vec{F} = m\vec{a}$ (Newton's second law of motion)

$$\underbrace{\vec{F}}_{\text{Force (N)}} = \underbrace{m\vec{a}}_{\text{rate of change of linear momentum}}$$

$$\frac{d}{dt}(m\vec{v})$$

F = Force (N)
m = mass (kg)
a = acceleration (m/s^2)

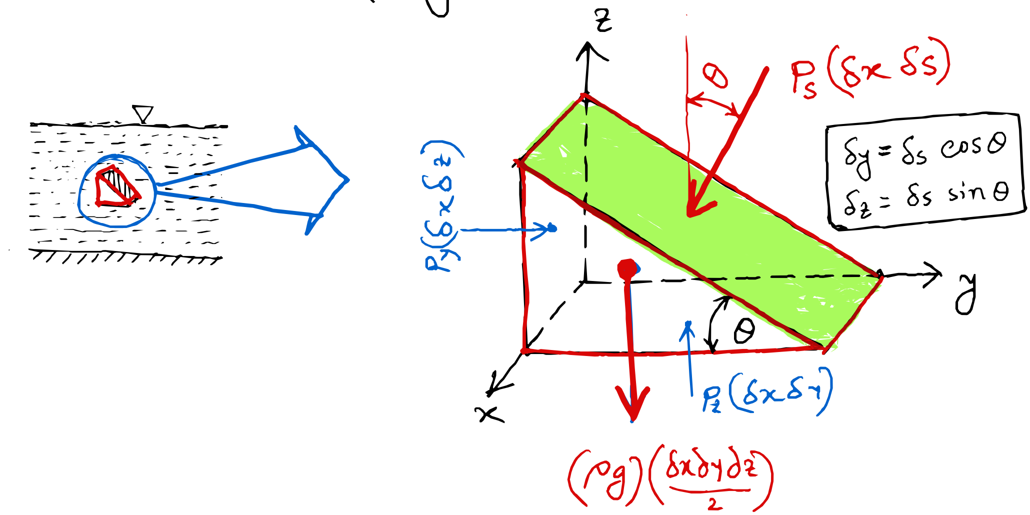
* what if more than one force is present?

$$\sum_{i=1}^{n} \vec{F}_i = m\vec{a}$$

("Above equation is a vector equation")

Pressure at a point

* Consider a wedge shaped fluid element as shown below. (Why wedge shape)



* y-direction

$$\sum F_i = m a_x \Rightarrow P_y(\delta x\,\delta z) - P_s(\delta x\,\delta s)\sin\theta = \left(\rho\,\frac{\delta x\,\delta y\,\delta z}{2}\right) a_y$$

$$\Rightarrow P_y(\delta x\,\delta s\sin\theta) - P_s(\delta x\,\delta s\sin\theta) = \rho\left(\frac{\delta x\,\delta y\,\delta z}{2}\right) a_y$$

$$\Rightarrow P_y - P_s = \frac{\rho}{2}\left(\frac{\cancel{\delta x}\,\delta s\cos\theta\cdot\cancel{\delta s\sin\theta}}{\cancel{\delta x}\,\cancel{\delta s}\,\cancel{\sin\theta}}\right) a_y$$

$$\Rightarrow P_y - P_s = (\delta s\cos\theta)\,\rho/2\; a_y$$

$$\Rightarrow P_y - P_s = \rho\, a_y\left(\frac{\delta y}{2}\right)$$

Since we are interested in pressure at a point we take the limit $\delta x \to 0$, $\delta y \to 0$ & $\delta z \to 0$.

$\therefore\ P_y = P_s$ (we did not assume stationary) z-direction: $\sum F_i = m a_z$

$$\Rightarrow P_z(\delta x \delta y) - P_s(\delta x \, \delta s \cos\theta) - \rho g \left(\frac{\delta x \delta y \delta z}{2}\right) = \rho \left(\frac{\delta x \delta y \delta z}{2}\right) a_z$$

$$\Rightarrow P_z - P_s = (\rho a_z + \rho g)\left(\frac{\delta z}{2}\right) \qquad \left[\text{using } \begin{array}{l} \delta y = \delta s \cos\theta \\ \delta z = \delta s \sin\theta \end{array}\right]$$

For a point $P_z = P_s$ (since $\delta z \to 0$)

* Pascal's law:

→ Pressure at a point (in a stationary or moving) of a fluid volume is "independent" of direction/orientation.

→ Pressure is a scalar (Always remmember)

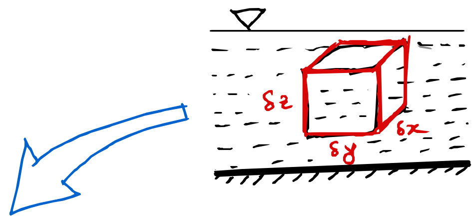
* Shear stress:

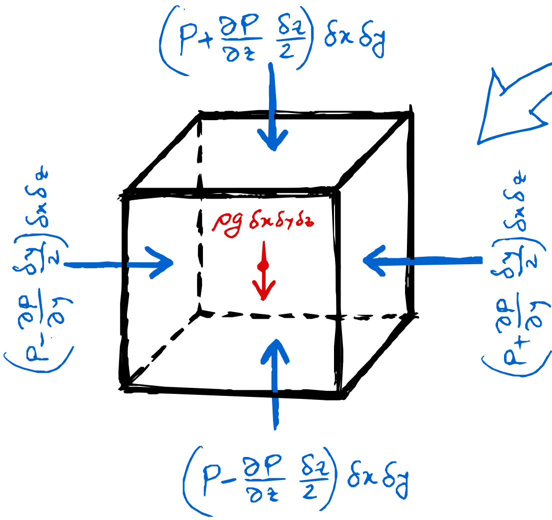
→ shear stress is a "tensor"!
(2nd-order tensor)

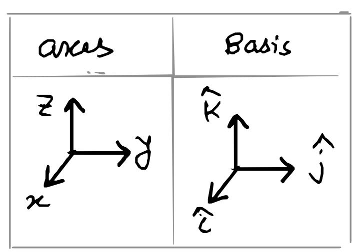
* Tensor:
  - 0 order tencor → scalar
  - 1 order tensor → vector
  - 2 order tensor → something else.

Equations for Pressure field

* Consider a fluid volume as shown in figure.







* Now apply Big Idea in vector form. (We need force components)

* $$\delta F_y = \left(P - \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z - \left(P + \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z$$

$$\Rightarrow \delta F_y = -\left(\frac{\partial P}{\partial y}\right)\delta x\,\delta y\,\delta z$$

* Similarly, $$\delta F_z = -\frac{\partial P}{\partial z}\,\delta x\,\delta y\,\delta z$$

$$\delta F_x = -\frac{\partial P}{\partial x}\,\delta x\,\delta y\,\delta z$$

(δF_x, δF_y and δF_z are surface forces)

Thus the force vector:

$$\delta \vec{F}_s = \delta F_x \hat{i} + \delta F_y \hat{j} + \delta F_z \hat{k}$$

$$\Rightarrow \delta \vec{F}_s = -\left[\left(\frac{\partial P}{\partial x}\right)\hat{i} + \left(\frac{\partial P}{\partial y}\right)\hat{j} + \left(\frac{\partial P}{\partial z}\right)\hat{k}\right] \delta x \delta y \delta z$$

$$\Rightarrow \delta \vec{F}_s = -\nabla P (\delta x \delta y \delta z)$$

* remmember the gradient operator?

$$\nabla = \left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}$$

$$\Rightarrow \left(\frac{\delta \vec{F}_s}{\delta x \delta y \delta z}\right) = -\nabla P$$

* Now the weight of the fluid volume

$$\delta \vec{W} = -(\rho g \, \delta x \delta y \delta z)\hat{k} \quad \text{(Body force)}$$

* Thus, $\sum \delta \vec{F} = \delta m \, \vec{a}$

$$\Rightarrow -\nabla P (\delta x \delta y \delta z) - \rho g \, \delta x \delta y \delta z \, \hat{k} = (\rho \, \delta x \delta y \delta z)\, \vec{a}$$

$$\Rightarrow -\nabla P - \rho g \, \hat{k} = \rho \vec{a}$$

* Considering $\vec{g} = g_x \hat{i} + g_y \hat{j} + g_z \hat{k}$ $(g_x = g_y = 0, \; g_z = -g)$

$$\Rightarrow \boxed{-\nabla P + \rho \vec{g} = \rho \vec{a}} \longleftarrow \text{(inertia)}$$

(Surface force) (Body force)

Pressure as a function of depth

* For fluid in rest $\vec{a} = 0$. Thus the vector equation for pressure field becomes

$$-\nabla P + \rho \vec{g} = 0 \qquad [\vec{g} = -g\hat{k}]$$

$$\Rightarrow -\left(\frac{\partial P}{\partial x}\right)\hat{i} - \left(\frac{\partial P}{\partial y}\right)\hat{j} - \left(\frac{\partial P}{\partial z}\right)\hat{k} - (\rho g)\hat{k} = 0$$

* isolating the components we obtain,

$$\frac{\partial P}{\partial x} = 0 \longrightarrow P \text{ not a function of } x$$

$$\frac{\partial P}{\partial y} = 0 \longrightarrow P \text{ not a function of } y$$

$$\frac{\partial P}{\partial z} = -\rho g \longrightarrow P \text{ is a function of } z.$$

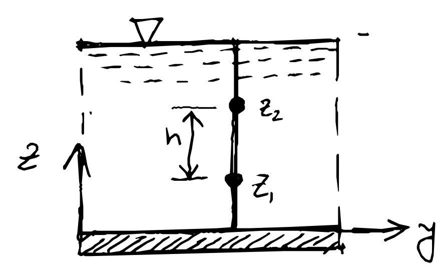
$$P = f(z)$$

* Integrating z component, in the range of

* $dP = -\rho g\, dz$

$$\Rightarrow \int_{P_1}^{P_2} = -\rho g \int_{z_1}^{z_2} dz$$

$$\Rightarrow P_2 - P_1 = -\rho g (z_2 - z_1)$$

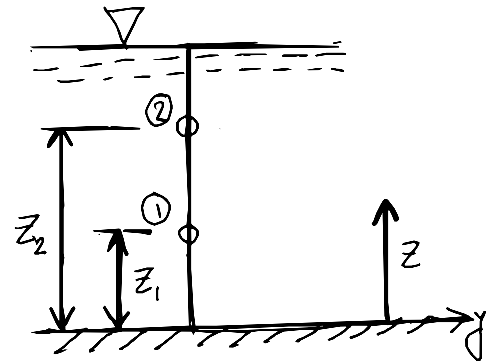


$(h = z_2 - z_1)$

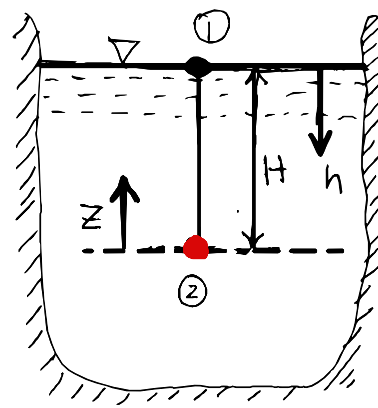
$\Rightarrow P_1 - P_2 = \rho g h$

$\Rightarrow P_1 = P_2 + \rho g h$

$\Rightarrow P_1 > P_2$



* Pressure at depth "H"



* $dP = -\rho g \, dz$

$\Rightarrow h = H - z$ (transformation)

$\Rightarrow dh = -dz$

$\therefore dP = \rho g \, dh$

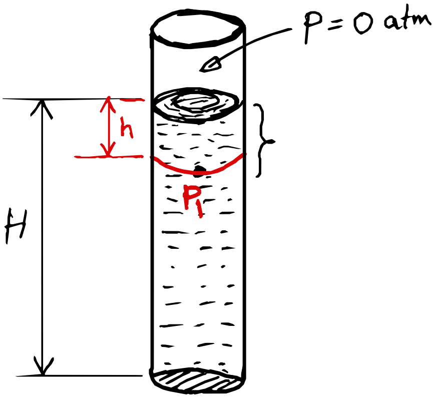
$\Rightarrow P_H - P_s = \rho g H$

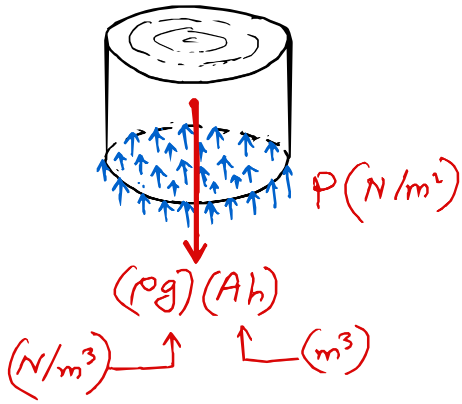
So the pressure at depth H can be written as $P_H = P_s + \rho g H$ ⟶ Increase in pressure.

P_s ⟶ pressure at free/top surface

* why P_h varies linearly with depth "h" ??

* Free-body diagram:





* Force on the interface at point P_1 is

$\downarrow$ Force = weight = density × volume

$= \rho g A H$

Area of force action = A

Big idea $\Rightarrow \sum F_i = 0$

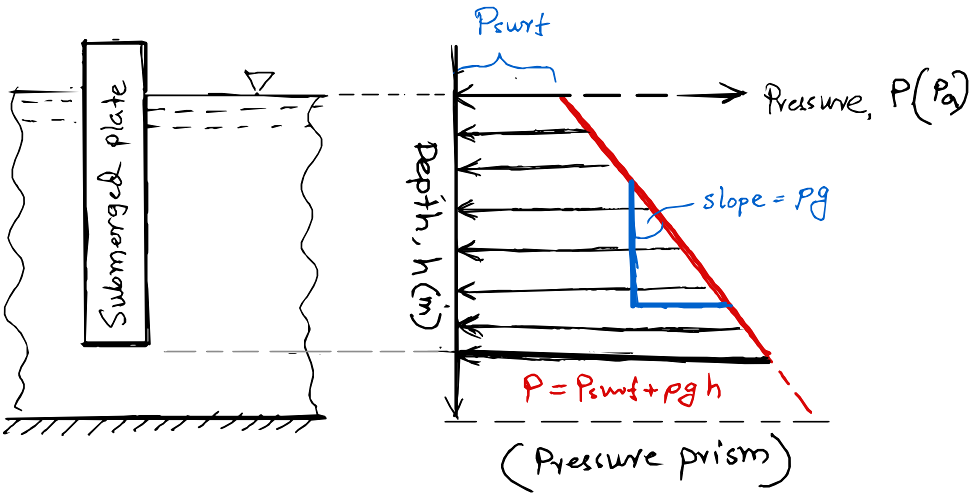
$\Rightarrow \rho g A H - P \cdot A = 0$

$\Rightarrow P = \rho g H$

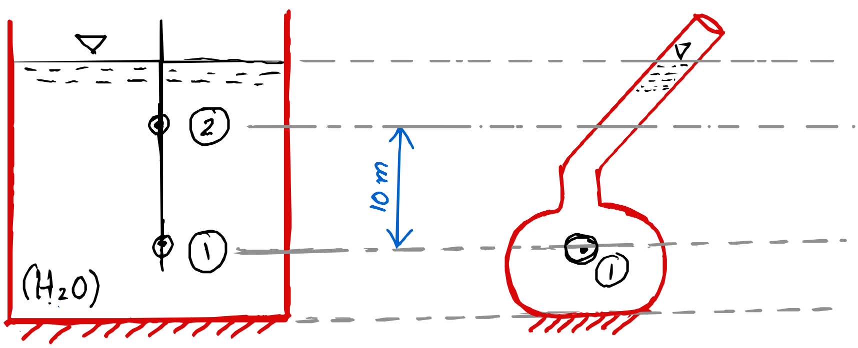
* If there is an additional pressure P_{surf} on top surface then pressure at depth 'h' is:

$$\boxed{P = P_{surf} + \rho g h}$$

* Pressure varies linearly with depth.



* Determine the pressure at point (2) if the pressure at point (1) is 2×10^5 Pa. (Both figures)



$P_2 = P_1 - \rho g h$

$\Rightarrow P_2 = 2 \times 10^5 - 10^3 \times 9.8 \times 10$

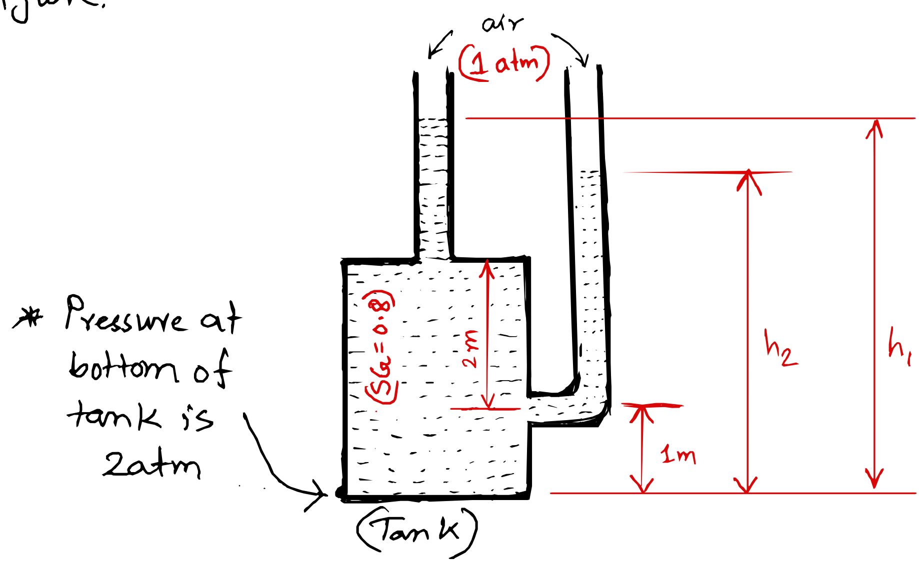
$\Rightarrow P_2 = 1.02 \times 10^5$ Pa

* Any thing different?

* What about amount of water above??

Practice

* A tank contains liquid oil of SG = 0.8. If the pressure at the tank bottom is found to be 2 atm determine the height h_1 & h_2 in the figure.



$\boxed{*}$ $$101325 = (2 \times 101325) - 0.8 \times 10^3 \times 9.8 \times h_1$$

$\boxed{*}$ $$101325 = (2 \times 101325) - 0.8 \times 10^3 \times 9.8 \times h_2$$

Thus $h_1 = h_2 = 12.92\ (m)$

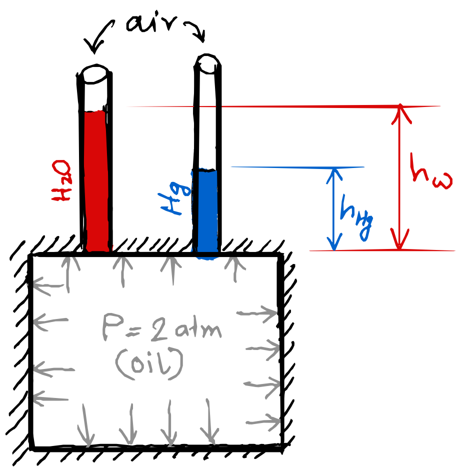
Ans

* Is it possible to calculate pressure if only the value of h is known for given fluid?

Pressure head

* Any pressure can be converted into liquid column height of some fluid. (Not necessarily same fluid as in the system)

* Determine the values of h_w and h_{Hg} in the figure below:



* $101325 = 2 \times 101325 - \rho_w g h_w$

$\Rightarrow h_w \approx 10.3 \text{ m}$

* $101325 = 2 \times 101325 - \rho_{Hg} g h_{Hg}$

$\Rightarrow h_{Hg} = 0.76 \text{ m}$

Thus, $\underbrace{10.3 \text{ m } H_2O}_{\text{water head corresponding to 1 atm}} \approx \underbrace{0.76 \text{ m } Hg}_{\text{Mercury head corresponding to 1 atm}}$

* Thus $1 \text{ atm} = 10.3 \text{ m } H_2O = 76 \text{ cm } Hg$.

* what is the blood pressure of normal human being? ⟶ 120/80 (unit ??)

* Ans: cm of Hg.

* Can we use a straight tube as a pressure measurement instrument?

ⓐ The fluid inside tank and tube must not mix or react.

ⓑ If more than needed amount of tube fluid is used it can get inside tank and contaminate the tank content.

ⓒ If water is used how long tube is needed to measure 1 atm pressure? (10.3 m ≈ 3 storied building)

* Remedy to each problem:

ⓐ Use immiscible/inactive fluid. (know the tank fluid)

ⓒ Use fluid with higher density (Hg).

(b) Bend the tube to U-shape to prevent accidental contamination.

* Determine h_1 & h_2

(a) $101325 = 2\times101325 - 13600\times9.8\times h_1$

$\Rightarrow h_1 = 76$ cm (known)

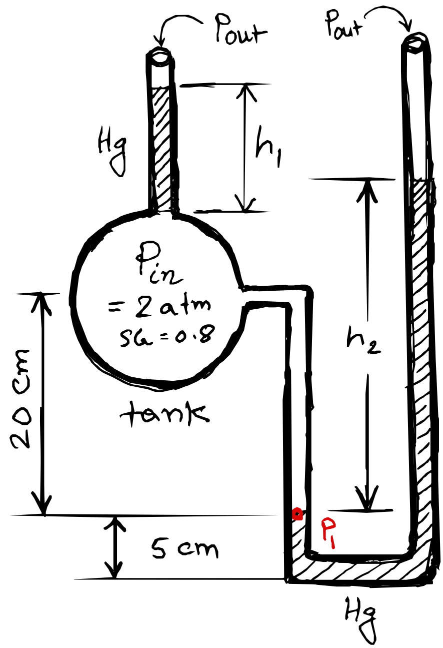
(b) Start at tank

* $2\times101325 + 800\times9.8\times0.2$

$+\cancel{(13600\times9.8\times0.05)}$

$-\cancel{(13600\times9.8\times0.05)}$

$-13600\times9.8\times h_2 = 101325$



$\Rightarrow 204218 - 133280\, h_2 = 101325$

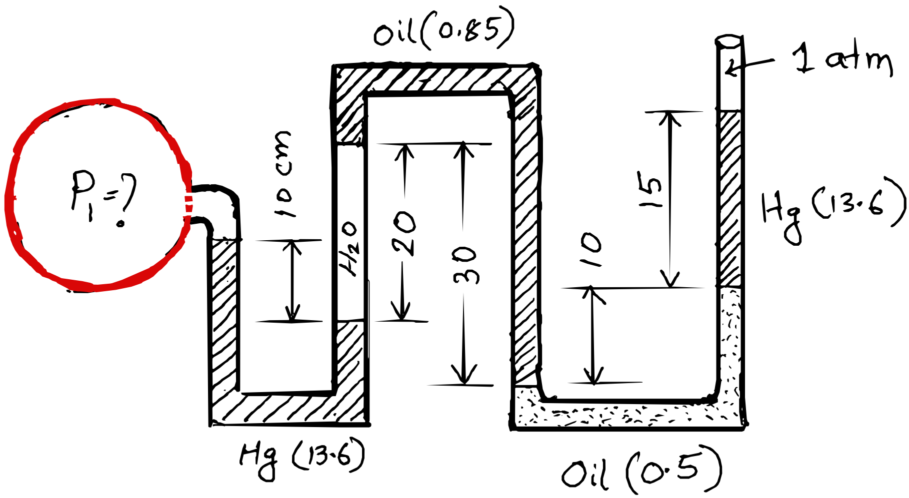
$\Rightarrow h_2 = 0.77$ m ≈ 77 cm (of Hg)

* Often the contribution of tank fluid (20 cm) is neglected by assuming that $P_1 = P_{in}$.

* Do we need to know the hight 5 cm? why not? explain.

Manometers

* Determine the pressure inside the tank.



(a) all distance are in cm
(b) Values in "( )" are specific gravity.

* $P_1 + 13600 \times 9.8 \times 0.1 - 1000 \times 9.8 \times 0.2 + 850 \times 9.8 \times 0.3$

$- 500 \times 9.8 \times 0.1 - 13600 \times 9.8 \times 0.15 = 101325$

$\Rightarrow P_1 - 6615 = 101325$

$\Rightarrow P_1 = 107940$ Pa ≈ 1.065 atm (Ans)

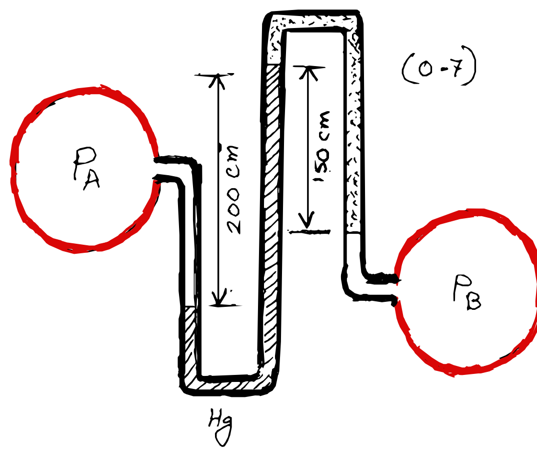
* what happens if the free end pressure is not 1 atm, but 3 atm.

$P_1 - 6615 = 3 \times 101325 \quad \therefore P_1 = 3.065$ atm

1. Manometer hights always represent pressure difference. Thus one end pressure must be known to determine other end pressure.

Differential Mannometer

* Determine pressure difference between two tanks in figure.



* $P_A - 13600 \times 9.8 \times 2 + 700 \times 9.8 \times 1.5 = P_B$

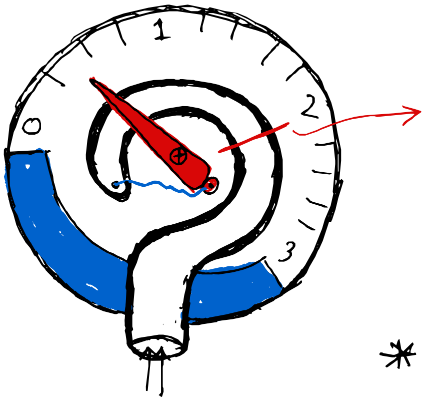
$\Rightarrow P_A - P_B = 256270 \approx \underline{2.53 \text{ atm}}$

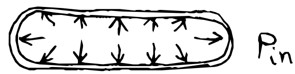
(Pressure difference)

Pressure measurement

① Manometer (Gives end pressure difference)

② Pressure Gauge (Gives gauge pressure)





cross Section of Bordon tube

(high Pressure fluid) P_{in}

* High pressure fluid makes the tube to try to be straight.

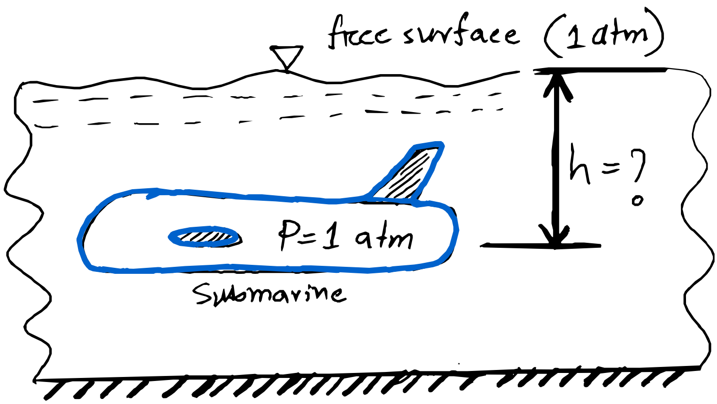
* when would Gauge show a reading of zero?? $P_{in} = P_{out}$.

* Reading is always positive.

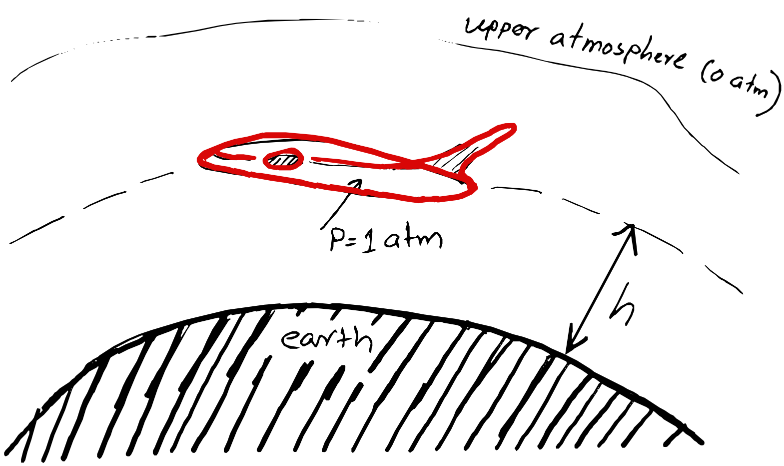
* (Absolute pressure) = (Gauge pressure) + (atmospheric pressure)

(up to this point we always talked about absolute pressure)

* Design a setup to measure dept of a submarine.

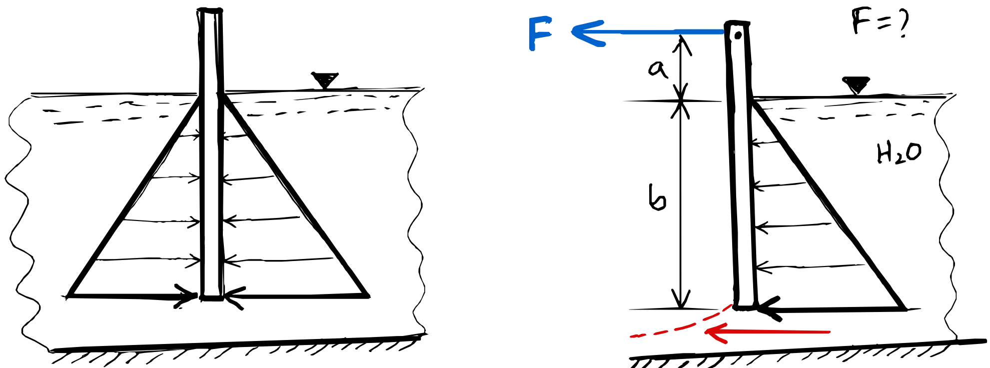


* Design a setup to measure elavation of an airplane.



Forces on Submerged plate

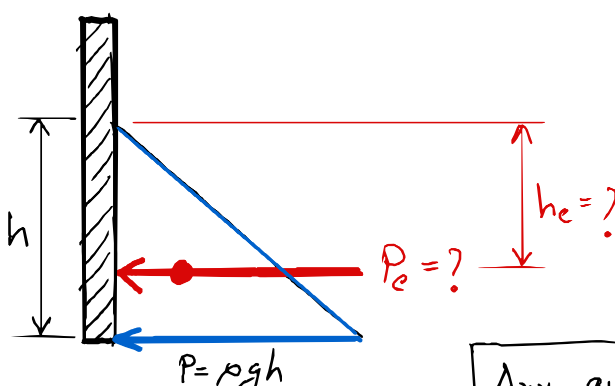
* Pressure varies linearly with dept (Pressure Prism)



(submerged plate) (sluice gate)

* Now we are talking about force/Torque not pressure alone.

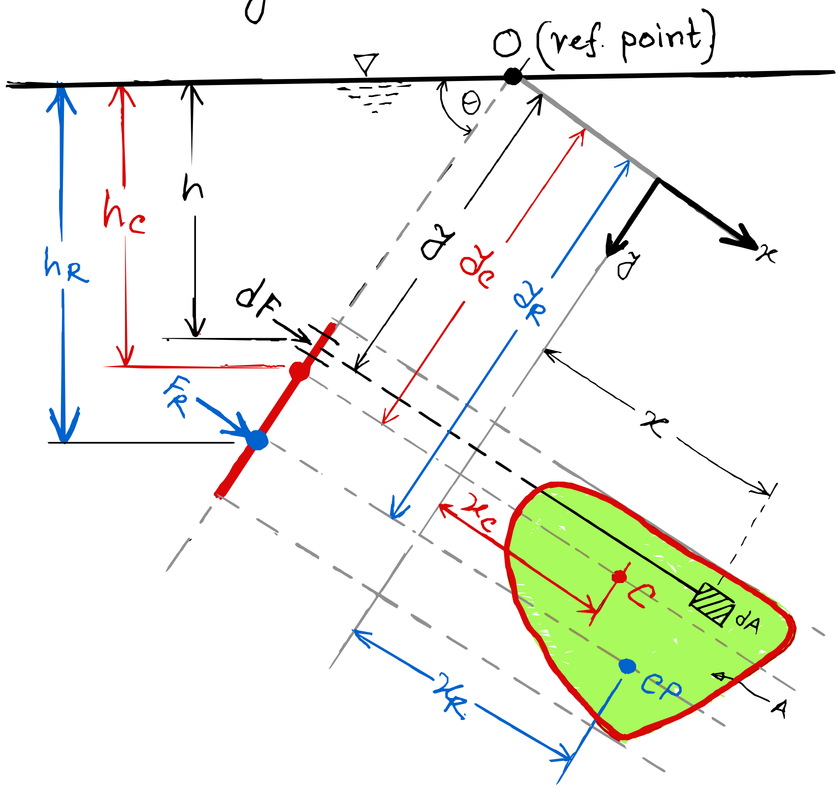
* where is the pressure center?



Any guess ??

Hydro-static force on plane surface

* Consider a plane (inclined) surface is fully immerged in a fluid (Density, ρ) as shown in the figure below.



* The equivalent force must reproduce (??) some parameter that is identical to the of the integrated local values.

* ⓐ Total force must remain same (F_R can be obtained from this)

(b) Moment must be same (with respect to the reference point O)

(location of y_R can be obtained)

* $F_R = \int_A dF = \int_A \rho g h \, dA = \int_A \rho g y \sin\theta \, dA$

$\Rightarrow F_R = \rho g \sin\theta \int_A y \, dA = \rho g \sin\theta \, (A \cdot y_c)$

Or, simply $\underline{\underline{F_R = (\rho g h_c) A}}$ (already known)

Resultant force = Pressure at area centroid × Submerged area of plate

* Moment about point "O":

$$F_R \, y_R = \int_A y \, dF = \int_A \rho g h y \, dA = \int_A \rho g y^2 \sin\theta \, dA$$

$$\Rightarrow F_R \, y_R = \rho g \cos\theta \int_A y^2 \, dA = \rho g \sin\theta \, I_x$$

$$\Rightarrow y_R = \left(\frac{\cancel{\rho g} \, \cancel{\sin\theta} \, I_x}{\cancel{\rho g} \, \cancel{\sin\theta} \, A y_c} \right) = \left(\frac{I_x}{A y_c} \right)$$

* Now from parallel axis theorem we know

$$I_x = I_{x_c} + A y_c^2$$

I_{x_c} are known For regular shapes see fig 2.18

Thus, $$\boxed{y_R = \left(\frac{I_{x_c}}{y_c A}\right) + y_c}$$

* For any plate, I_{x_c}, A is positive. y_c is also positive when submerged fully. Thus

$y_R > y_c$

$\Rightarrow \boxed{h_R > h_c}$

(Depth of centroid of the submerged plate area)

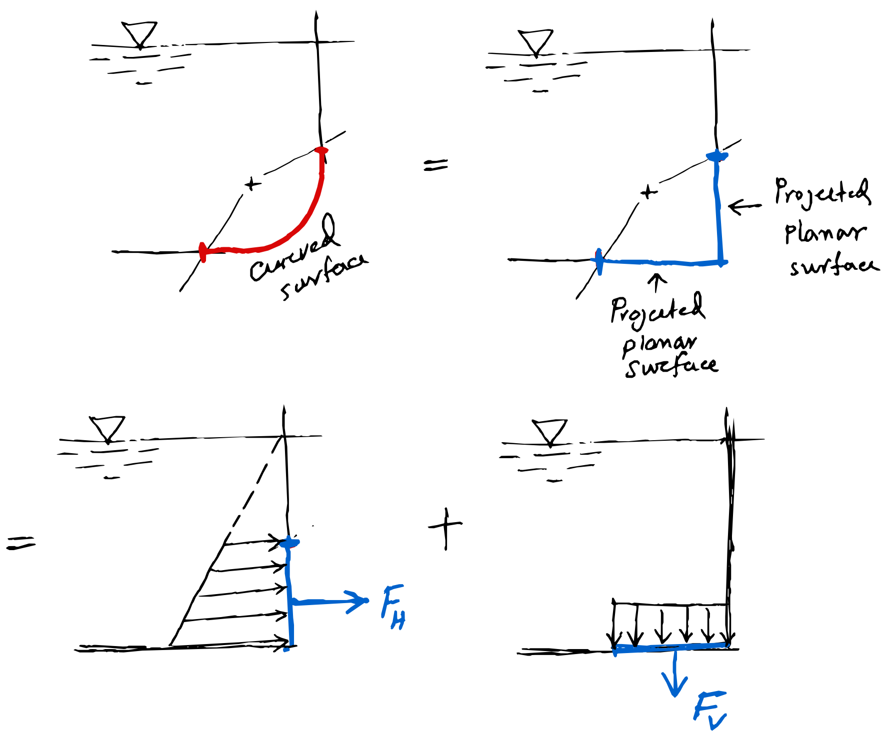
Depth of location of equivalent/Resultant force action. (Depth of pressure center)

ⓐ why $h_R > h_c$?

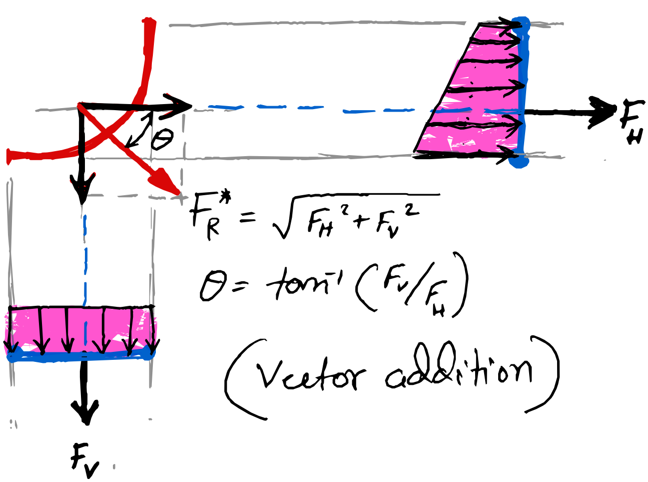
ⓑ Alway true!!

* Similar analysis can be done to obtain x_R as well. Not very important. (Comes as function of polar moment of inertia)

Hydostatic force on a curved surface



Final superposition



$$F_R^* = \sqrt{F_H^2 + F_V^2}$$

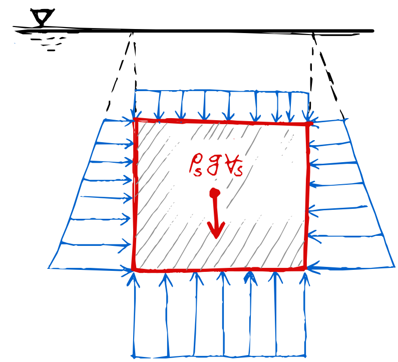
$$\theta = \tan^{-1}(F_V / F_H)$$

(Vector addition)

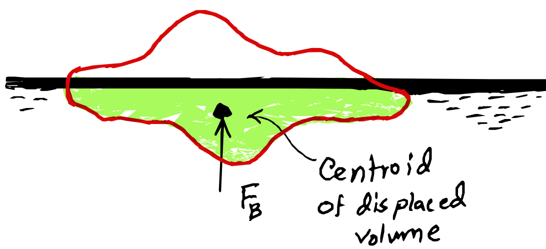
Buoyancy

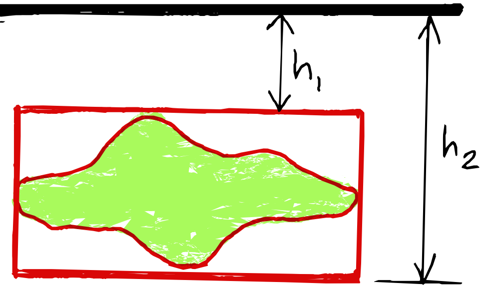
* Pressure gradient causes upward force

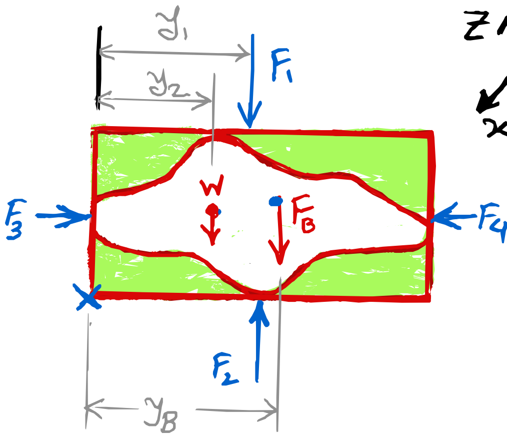
* Mass/weight causes downward force

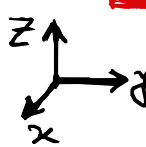


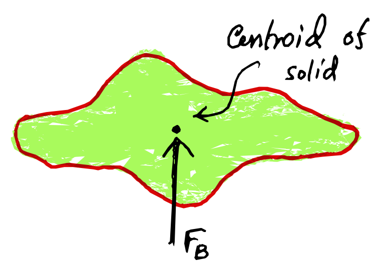
Force & moment











* To find F_B and y_B we apply force and moment balance. (Equilibrium) on the fluid volume shown.

$$\sum F = 0 \qquad \Rightarrow F_B = F_2 - F_1 - W$$

* from hydrostatic pressure, $F_2 - F_1 = \rho g (h_2 - h_1) A$

Thus, $F_B = \rho g (h_2 - h_1) A - W \longrightarrow$ (How to obtain this ??)

$\longrightarrow$ {It is the weight of not displaced volume.

$$W = \rho g \forall_{undisp}$$

$$\Rightarrow W = \rho g \left[(h_2 - h_1) A - \forall \right]$$

$\longrightarrow \forall$ volume of displaced fluid.

$$\Rightarrow W = \rho g (h_2 - h_1) A - \rho g \forall$$

* $F_B = \cancel{\rho g (h_2 - h_1) A} - \cancel{\rho g (h_2 - h_1) A} + \rho g \forall = \rho g \forall$

$$\boxed{F_B = \underbrace{\rho g \forall}_{}}$$

* weight of displaced fluid volume

* Moment about point "x" (see fig)

$$F_B \, y_B = F_2 y_1 - F_1 y_1 - W y_2$$

substitution of all forces gives

$$\cancel{\rho g} \forall y_B = \cancel{\rho g} h_2 A y_1 - \cancel{\rho g} h_1 A y_1 - \cancel{\rho g} (h_2 - h_1) A y_2 + \rho g \forall y_2$$

* Thus $\forall \, y_B = \forall_T \, y_1 - (\forall_T - \forall) \, y_2$

$$\Rightarrow y_B = \left(\frac{\cancel{\forall_T} \, y_1 - \forall_T \, y_2}{\forall} \right)$$

$\Rightarrow y_B =$ Centroid of displaced volume $\forall$.

* Buoyancy force always act upward (surface) and always acts on the centroid of the displaced fluid volume.

Some Comments

1. Buoyancy force is equal to the weight of the displaced fluid volume

   $F_B = \rho g \forall$ (Archimedes's principle)

2. F_B always act in opposite direction to the gravity (upward).

3. F_B always goes through the centroid of displaced fluid volume $\forall$.

4. F_B reduces the weight of the submerged solid (known as apparent weight)

Floatation

* If $F_B < W$ the solid sinks. (Case-1)
  → The solid weight (Not mass) decreases.
* If $F_B < W$ the solid floats. (Case-2)
  → The weight of liquid displaced is equal to the weight of solid.

Case-1 (Sink): While sinking in water, a solid (SG = 1.2) would loose what % of its weight?

Solution: Volume of solid = $\forall_s$

weight of solid $= (1.2 \times 10^3 \times 9.8)\forall_s$

$\Rightarrow W = 11760 \forall_s$ (N)

Buoyancy force, $F_B = 10^3 \times 9.8 \times \forall_s$

$\Rightarrow F_B = 9800 \forall_s$ ($<W$)

weight lost $= F_B = 9800 \forall_s$

$\therefore$ % weight lost $= \left(\dfrac{9800 \forall_s}{11760 \forall_s}\right) \times 100\ \%$

$= 83.3\ \%$

* Water is lighter than the solid. (by what %)

$(\rho_{H_2O}/\rho_{solid}) \times 100\% \simeq 83.3\ \%$

(coincidence !!)

* Case-2 (float): while floating what % of a solid ($SG = 0.91$) remains above the water?

Solution: Volume of the solid $= \forall_s$

weight of the solid, $W = 8918 \forall_s$ (N)

Buoyancy force, $F_B = 9800 \forall_s$ ($> W$)

* Assume the volume of water displaced $= \forall$

Thus, $\begin{pmatrix}\text{weight of} \\ \text{the solid}\end{pmatrix} = \begin{pmatrix}\text{weight of} \\ \text{water displaced}\end{pmatrix}$

$\Rightarrow \quad W = \rho_{H_2O} \cdot g \cdot \forall$

$\Rightarrow \quad 8918 \forall_s = 9800 \forall$

$\Rightarrow \quad \forall = \underline{\underline{0.91}} \forall_s$ ⟶ Coincidence!!

* Volume % above water $= \left(\dfrac{\forall_s - \forall}{\forall_s} \right) \times 100\%$

$\approx 9\%$

* What solid has $SG = 0.91$? (ice)

* Practice: while floating in liquid Hg, a solid retains 1% of its volume above the free surface. Determine the SG of the solid.

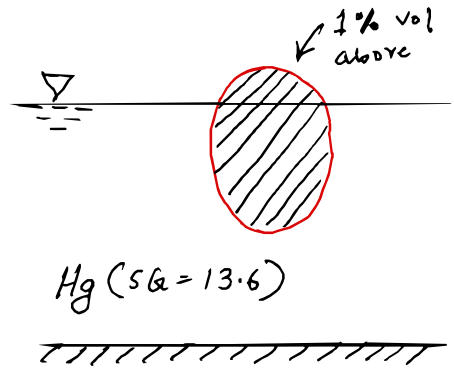
Solution

Assume,

* Total volume of solid = $\forall_s$

* weight of solid $W = (SG) \times 9800 \, \forall_s$

* Displaced liquid volume, $\forall = 0.99 \, \forall_s$

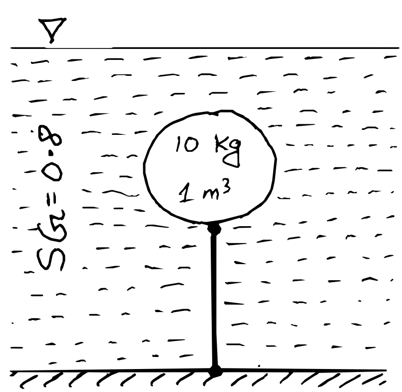


→ For floating, $W = \rho_{Hg} \, g \, \forall$

$$\Rightarrow (SG) \times 9800 \, \forall_s = 13600 \times 9.8 \times 0.99 \, \forall_s$$

$$\Rightarrow (SG) = 13.46 \quad \text{(Ans)}$$

* Determine the tension in cable for the under water float.



* $F_B = 800 \times 9.8 \times 1 \, (N)$
  $= 7840 \, (N)$

* $W = 10 \times 9.8 \, (N)$
  $= 98 \, (N)$

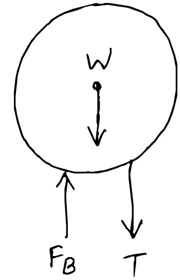
* Free body diagram: (Big idea)

Thus, $T = F_B - W$

$\Rightarrow T = 7840 - 98$ (N)

$\Rightarrow T \approx 7742$ (N)

(Ans)



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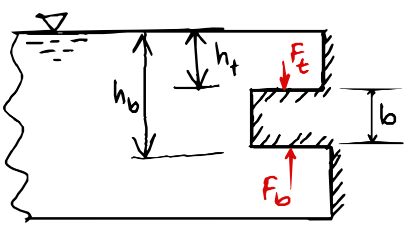
* Check that Archimedes's principle hold's for "Combined" shape too.



* Net upward force

$= (F_b - F_t) = \rho g (h_b - h_t) A$

$= \rho g b A = \rho g \forall$

(displaced liquid volume)

* Conclusion

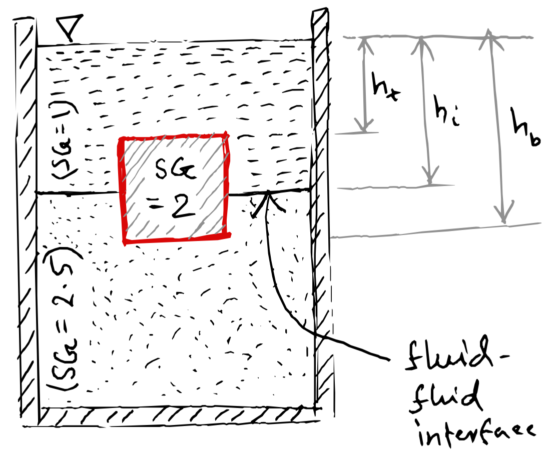
$\longrightarrow$ when fluid is displaced, sorrounding fluids creates an upward (against gravity) force equal to the weight of displaced fluid volume.

Floating and Sinking Simultaneously

* If a solid cube (SG = 2) is released in a glass containing two immiscible liquids (SG = 1 and 2.5) what fraction of solid will sink in bottom fluid?

* Pressure at top

$$P_t = 9800\, h_t$$



* Pressure at bottom

$$P_b = 9800\, h_i + 2.5 \times 9800\,(h_b - h_i)$$

* Net upward forc

$$F_B = (P_b - P_t)\,A = 9800\, h_i + 24500\, h_b - 24500\, h_i - 9800\, h_t$$

$$= 24500\, h_b - 9800\, h_t - 14700\, h_i$$

* Weight, $W = 2 \times 9800\, \forall_s = 19600\,(h_b - h_t) A$

* Now $W = F_B$ (Big idea)

$$\Rightarrow 19600\,h_b - 19600\,h_t = 24500\,h_b - 9800\,h_t - 14700\,h_i$$

$$\Rightarrow +4900\,h_b + 9800\,h_t = +14700\,h_i$$

$$\Rightarrow 4900\,h_b - 4900\,h_t = 14700\,h_i - 9800\,h_t - 4900\,h_t$$

$$\Rightarrow 4900\,(h_b - h_t) = \underline{14700\,h_i} - \underline{14700\,h_t}$$

same coefficient !!

coincidence ??

$$\Rightarrow 4900\,(h_b - h_t) = 14700\,(h_i - h_t)$$

$$\Rightarrow \left(\frac{h_i - h_t}{h_b - h_t}\right) = \left(\frac{4900}{14700}\right)$$

$$\Rightarrow \frac{(h_i - h_t)A}{(h_b - h_t)A} = \left(\frac{1}{3}\right)$$

$$\Rightarrow \left(\frac{V_{top}}{V_{tot}}\right) = \left(\frac{1}{3}\right)$$

* 1/3 portion of the solid cube will be above the interface, while $\frac{2}{3}$ portion will be below the interface.

* What is Buoyancy ?

→ Up ward force (due to pressure gradient)

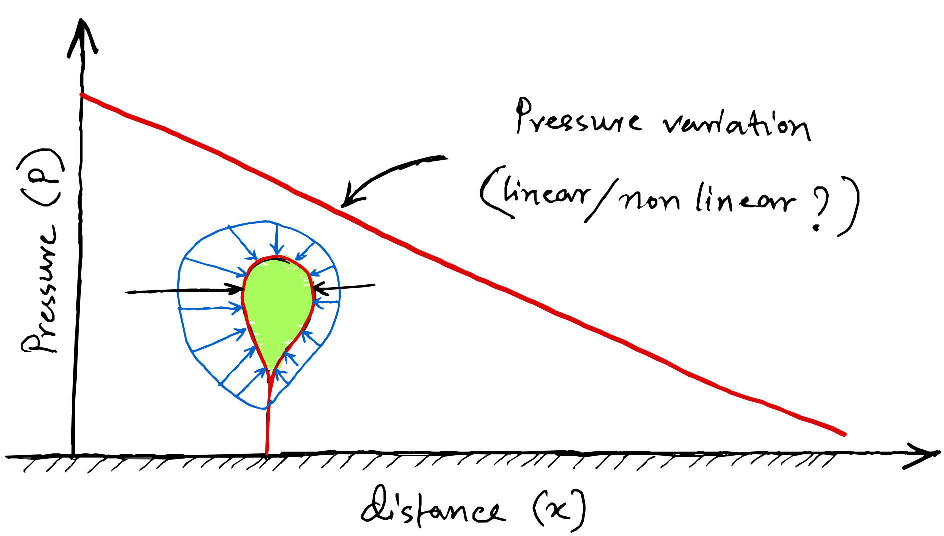
→ always !!

→ More accurate is to say along the direction of pressure gradient.

* Example : See the video on youtube (V-3)

→ The acceleration of the car causes the air pressure to go up at the back of the car. (due to inertia)

→ The baloon is pushed forward.



Rigid-body motion

* why in statics? (Not dynamics)
* what is rigid-body motion?

* The fundamental equation for pressure field

$$\underbrace{-\nabla P}_{\text{Surface force}} + \underbrace{\rho \vec{g}}_{\text{Body force}} = \underbrace{\rho \vec{a}}_{\text{Inertia}}$$

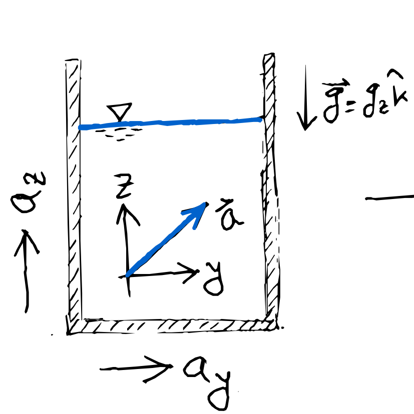
* The vector equation is basically 3 scalar equations.

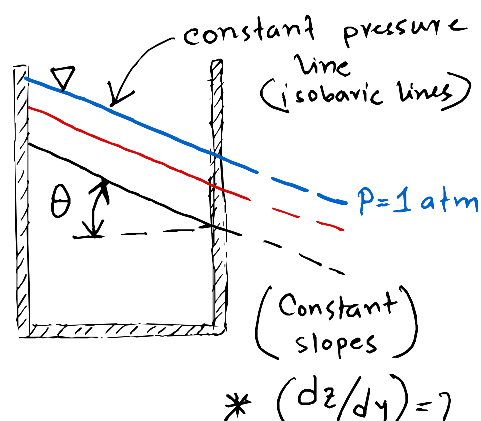
ⓐ $-\dfrac{\partial P}{\partial x} + \rho g_x = \rho a_x \longrightarrow \dfrac{\partial P}{\partial x} = -\rho a_x \quad (g_x = 0)$

ⓑ $-\dfrac{\partial P}{\partial y} + \rho g_y = \rho a_y \longrightarrow \dfrac{\partial P}{\partial y} = -\rho a_y \quad (g_y = 0)$

ⓒ $-\dfrac{\partial P}{\partial z} + \rho g_z = \rho a_z \longrightarrow \dfrac{\partial P}{\partial z} = +\rho g - \rho a_y \quad (g_z = -g)$

* linear motion with constant acceleration.





* $(dz/dy) = ?$

* Pressure field: $\frac{\partial P}{\partial x} = 0$ $\boxed{P \neq f(x)}$

$\frac{\partial P}{\partial y} = -\rho a_y$ $\boxed{P = f(y)}$

$\frac{\partial P}{\partial z} = -\rho g - \rho a_z$ $\boxed{P = f(z)}$

* Pressure differece can be obtained from these slopes as

$$dP = \cancelto{0}{\left(\frac{\partial P}{\partial x}\right)} dx + \left(\frac{\partial P}{\partial y}\right) dy + \left(\frac{\partial P}{\partial z}\right) dz$$

$$\Rightarrow dP = \left(\frac{\partial P}{\partial y}\right) dy + \left(\frac{\partial P}{\partial z}\right) dz$$

$$\Rightarrow dP = -\rho a_y \, dy - \rho g \, dz - \rho a_z \, dz$$

$$\Rightarrow dP = -(\rho a_y) dy - \rho (g + a_z) dz$$

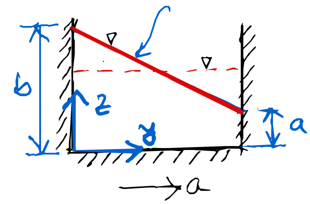
* For constant pressure lines $dP = 0$

thus, $$\boxed{\left(\frac{dz}{dy}\right) = -\left(\frac{a_y}{g + a_z}\right)}$$ $\left(\begin{matrix} g\downarrow & a_y \rightarrow \\ a_z \uparrow & \end{matrix} \right)$

(Slope we wanted to know)

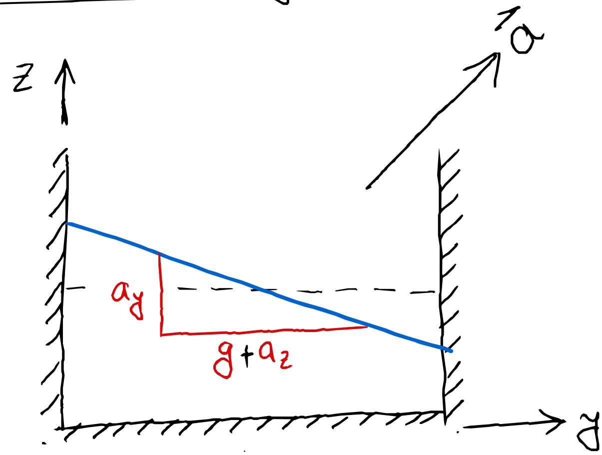
Do yourself

* Find the equation of the free surface. (e.g. value of "a" or "b")



* What happens when (a) $a_y = 0$, $a_z \neq 0$ (b) $g = 0$ (c) $a = -g_z$ (d) $a_z \rightarrow \infty$

* Slope investigation



slope

$$\left(\frac{dz}{dy}\right) = -\left(\frac{a_y}{g+a_z}\right)$$

ⓐ if $a_y = 0$ (vertical acceleration only), then constant pressure lines remains horizontal.

* Fluid weight increases (a_z +ve) or decreases (a_z −ve). ($W_{new} = \rho(g+a_z)\forall$)

ⓑ if $a_z = 0$ (Horizontal acceleration only), then the slope is determined by g and a_y.

* No weight change is achieved.

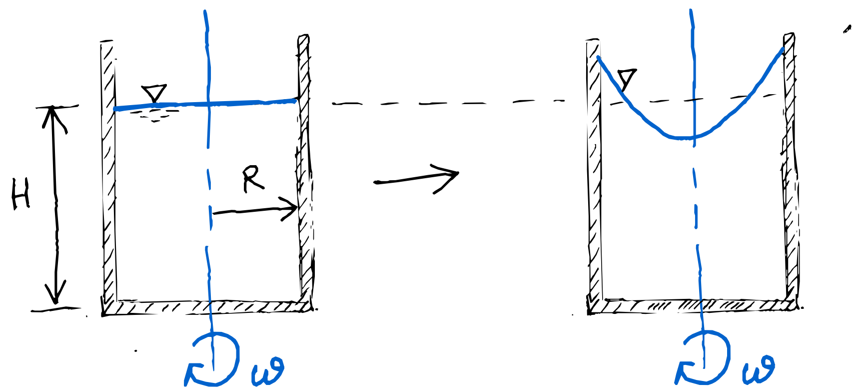
ⓒ if $a_z = -g$, the fluid becomes weightless.

* When $g = 0$ (space), do we need to do any work to accelerate a volume of water vertically ?? ⟶ we do

→ If yes, then why ?

Rigid-body rotation

* Rotation with constant angular velocity.



* Use cylindrical co-ordinate (r, θ, z)

$$\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$ (Try to memorize)

* acceleration vector $\vec{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta + a_z\hat{e}_z$

where, $a_r = -r\omega^2$

$a_\theta = 0$

$a_z = 0$

(a) where is this $a_r \neq 0$ coming from?

(b) Does velocity $\vec{v}$ (v_r, v_θ and v_z) changes with time?

* Basic equation:

$$-\nabla P + \rho\vec{g} = \rho\vec{a}$$ → Valid for all co-ordinate systems!

* 3-scalar equations thus,

$$\frac{\partial P}{\partial r} = \rho \omega^2 r \qquad \boxed{P = f(r)}$$

$$\frac{\partial P}{\partial \theta} = 0 \qquad \boxed{P \neq f(\theta)}$$

$$\frac{\partial P}{\partial z} = -\rho g \qquad \boxed{P = f(z)}$$

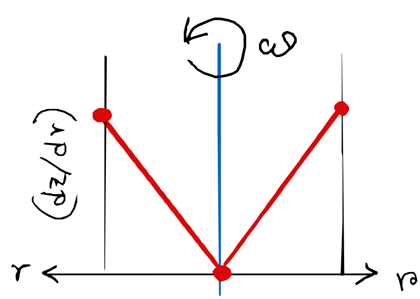
$$\Rightarrow dP = \left(\frac{\partial P}{\partial r}\right) dr + \cancelto{0}{\left(\frac{\partial P}{\partial \theta}\right)} d\theta + \left(\frac{\partial P}{\partial z}\right) dz$$

$$\Rightarrow dP = (\rho \omega^2 r)\, dr - \rho g\, dz$$

* For constant pressure line $dP = 0$

Thus, $$\frac{dz}{dr} = \left(\frac{\rho \omega^2 r}{\rho g}\right) = \left(\frac{\omega^2}{g}\right) r$$

* Slope of constant pressure line is not constant.
* Slope increases linearly with radial distance.



* Integrate the slope equation to obtain free-surface equation.

* $\dfrac{dz}{dr} = \left(\dfrac{\omega^2}{g}\right) r \quad \Rightarrow \quad dz = \left(\dfrac{\omega^2}{g}\right) r\,dr$

$\Rightarrow \quad z = \left(\dfrac{\omega^2 r^2}{2g}\right) + C \qquad (C = \text{constant})$

* How to obtain constant C ??

* What other constrain (physical law) must be satisfied ?

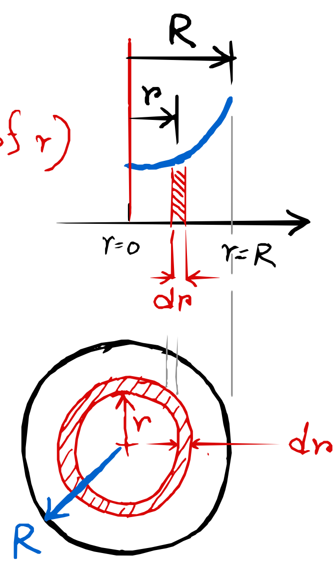
  → Conservation of mass. (ok)

  → Conservation of volume !!

  ↳ * (Only apply to incompressible fluids)

* If mass of the fluid is m and the volume is $\forall$ then, $m = \rho \forall$, must not change.

* Initial mass $= \rho(\pi R^2) H$



(function of r)

* Final mass $= \displaystyle\int_0^R \rho\, Z\, 2\pi r\, dr$

$= 2\pi\rho \displaystyle\int_0^R r\left(\dfrac{\omega^2 r^2}{2g}\right) + Cr\, dr$

$= \dfrac{\pi\rho\omega^2}{g}\left(\dfrac{R^4}{4}\right) + \pi\rho R^2 C$

* From mass balance,

$$\frac{\pi\rho\omega^2 R^4}{4g} + (\pi\rho R^2)C = \pi\rho R^2 H$$

$$\Rightarrow C\,(\cancel{\pi\rho R^2}) = (\cancel{\pi\rho R^2})\left[H - \left(\frac{\omega^2 R^2}{4g}\right)\right]$$

$$\Rightarrow C = \left[H - \left(\frac{\omega^2 R^2}{4g}\right)\right]$$

Thus,

$$Z = \frac{\omega^2 r^2}{2g} + H - \frac{\omega^2 R^2}{4g}$$

$$\Rightarrow Z = H + \left(\frac{\omega^2 R^2}{2g}\right)\left[(r/R)^2 - \frac{1}{2}\right]$$

* Dip at center $(r=0)$

$$\text{dip} = H - Z_{r=0} = H - \left[H - \frac{\omega^2 R^2}{4g}\right]$$

$$\Rightarrow \text{dip} = \left(\frac{\omega^2 R^2}{4g}\right)$$

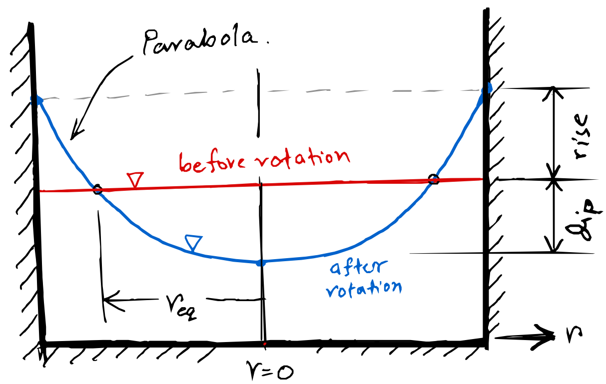
* Rise at edge $(r=R)$

$$\text{rise} = Z_{r=R} - H = H + \left(\frac{\omega^2 R^2}{4g}\right) - H$$

$$\Rightarrow \text{rise} = \left(\frac{\omega^2 R^2}{4g}\right)$$

* Dip at center = Rise at edge !!

*

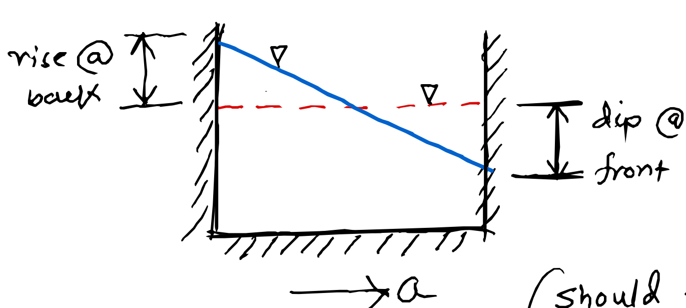


Do yourself

① Find the expression for r_{eq} (radial location when hight of free surface does not change)

$\left(r_{eq} = R/\sqrt{2}\right)$ → why does not depend on ω?

② For rigid-body acceleration does the dip at front wall and rise as back wall equal?



(should not be equal??)

Compressible fluids

* For compressible fluid density can change.
  → Can we use $p = \rho g h$?? yes !!

* Before using $p = \rho g h$ we need to remmber that $\rho = f(P, T)$

  → Density is dependent on Pressure & Temp.

$$P = \rho R T \longrightarrow \text{(ideal gas model)}$$

* Now we focus on pressure changes as function of dept (z-axis)

$$dp = \rho g dz$$

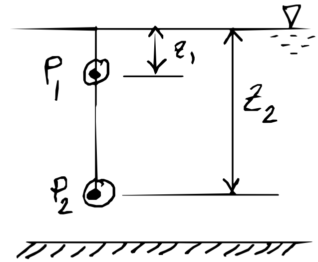
$$\Rightarrow dp = \left(\frac{P}{RT}\right) g dz$$

$$\Rightarrow \frac{dP}{P} = \left(\frac{g}{RT}\right) dz$$

$$\Rightarrow \int \frac{dP}{P} = \int \frac{g}{RT} dz$$

* Does 'T' changes with z ? If yes then how!
* We assume that T does not changes with z.

* Using the proper limits



$$\int_{P_1}^{P_2} \frac{dP}{P} = \frac{g}{RT_0} \int_{z_1}^{z_2} dz$$

$$\Rightarrow \left[\ln P\right]_{P_1}^{P_2} = \frac{g(z_2 - z_1)}{RT_0}$$

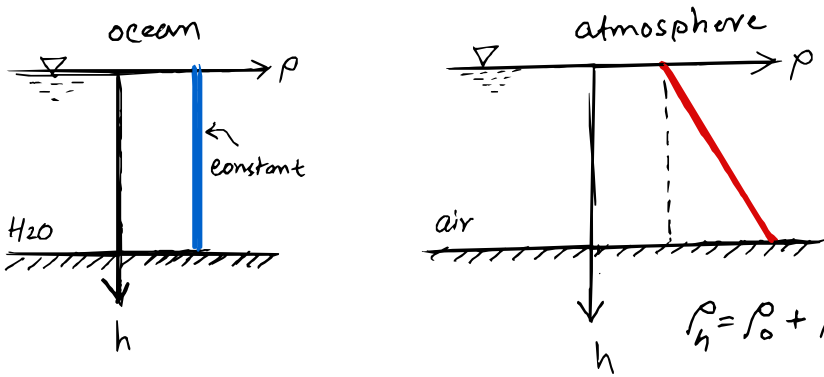
$$\Rightarrow \left(P_2/P_1\right) = \exp\left[\frac{g(z_2 - z_1)}{RT_0}\right]$$

$$\Rightarrow \quad P_2 = P_1 \exp\left[gh/RT_0\right]$$

* Pressure distribution become nonlinear and more specifically exponential.

For demonstration only

* Pressure variation for linearly increasing density of fluid.



$$\rho_h = \rho_0 + \beta h$$

* Pressure, $P = \rho_h g h$

$$\Rightarrow dP = \rho_h g \, dh \qquad \boxed{\rho = f(h)}$$

$$\Rightarrow dP = \rho_0 g \, dh + \beta g h \, dh$$

$$\Rightarrow \int_{P_0}^{P} dP = \rho_0 g \int_0^h dh + \beta g \int_0^h h \, dh$$

$$\Rightarrow P = P_0 + \rho_0 g h + \beta g h^2/2$$