ME 247
(Engineering Mechanics-Statics)

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Moment of Inertia of an Area

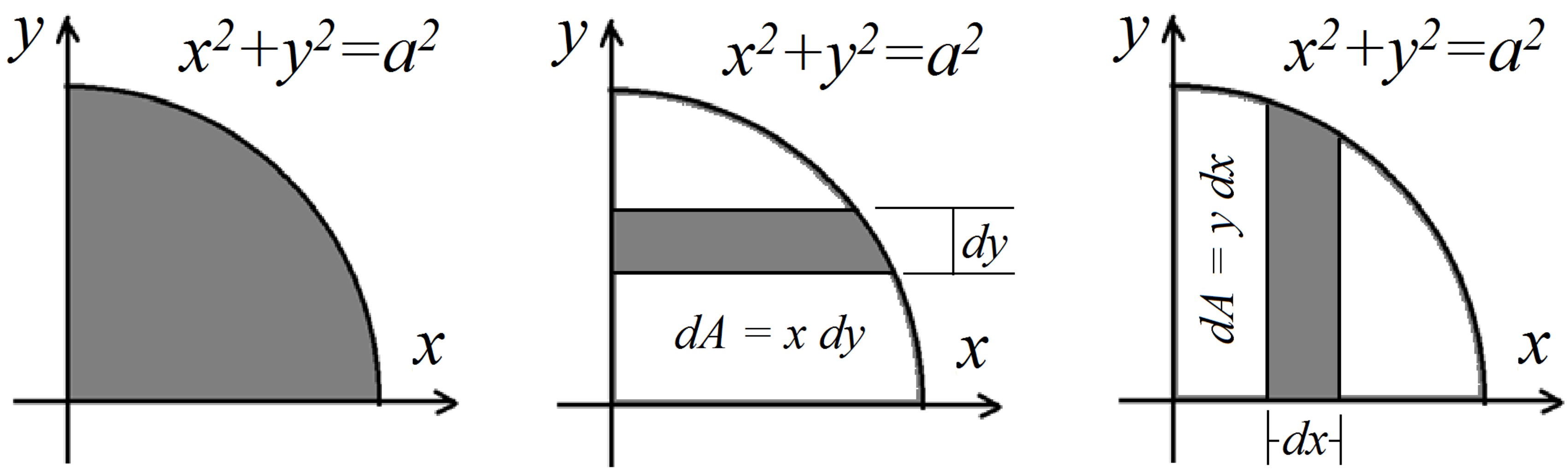
$$I_x = \int y^2 dA \qquad\qquad I_y = \int x^2 dA$$

** Here '*dA*' is an elementary area selected parallel to the reference axis.

*** Here *x* and *y* are two parameters that represents the distance of '*dA*' from reference axis

Problems (Moment of Inertia)

Example-1: Find the Moment of Inertia for the triangular area shown.

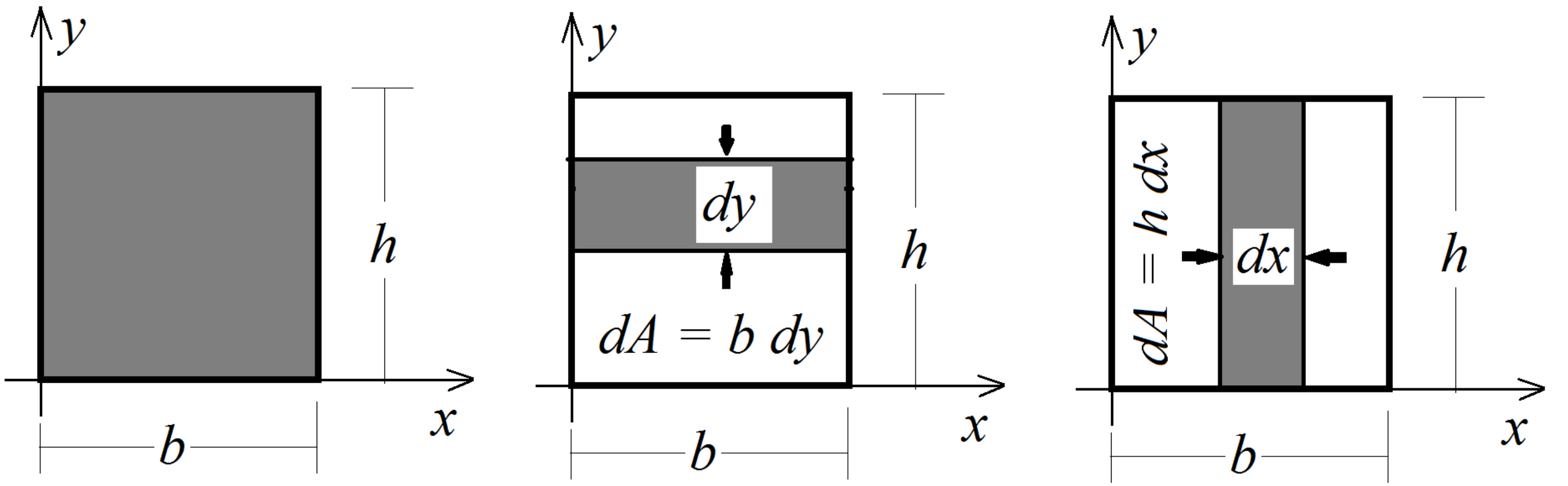


$$I_x = \int y^2\, dA = \int y^2\, x\, dy = \int_0^a y^2 \sqrt{a^2 - y^2}\; dy$$

$$I_y = \int x^2\, dA = \int x^2\, y\, dx = \int_0^a x^2 \sqrt{a^2 - x^2}\; dx$$

Problems (Moment of Inertia)

Example-2 (a): Find the Moment of Inertia for the area shown.

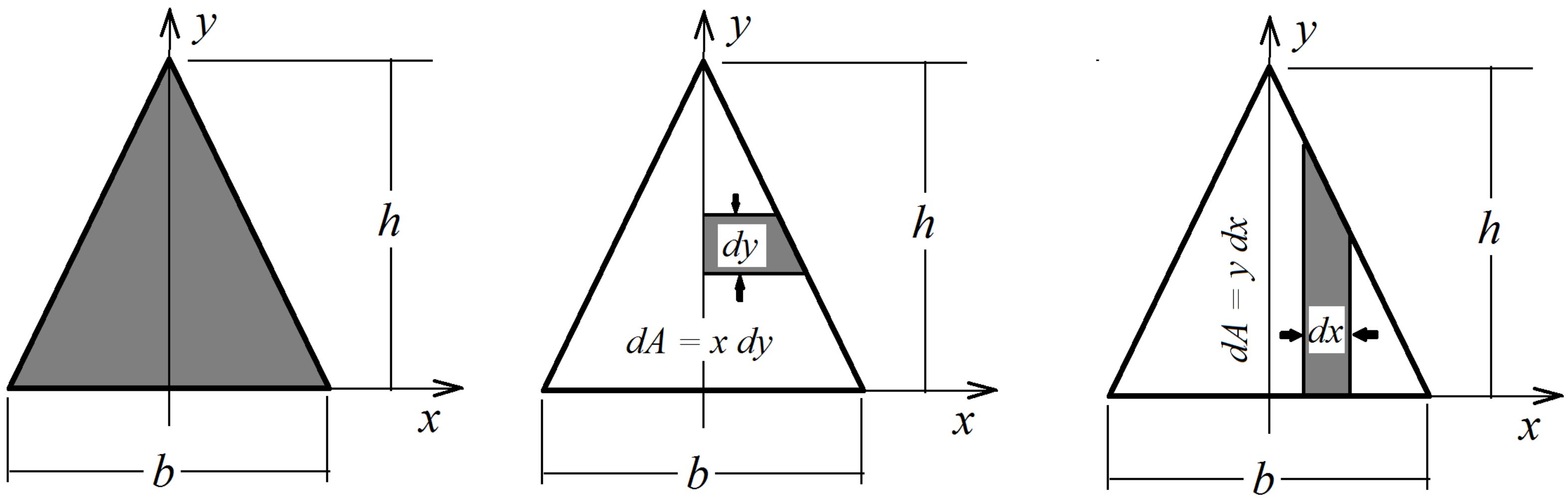


$$I_x = \int y^2\, dA = \int_0^h y^2\, b\, dy = \frac{1}{3}\, bh^3 \quad (unit)^4$$

$$I_y = \int x^2\, dA = \int_0^b x^2\, h\, dx = \frac{1}{3}\, b^3 h \quad (unit)^4$$

Problems (Moment of Inertia)

Example-2(b): Find the Moment of Inertia for the triangular area shown.

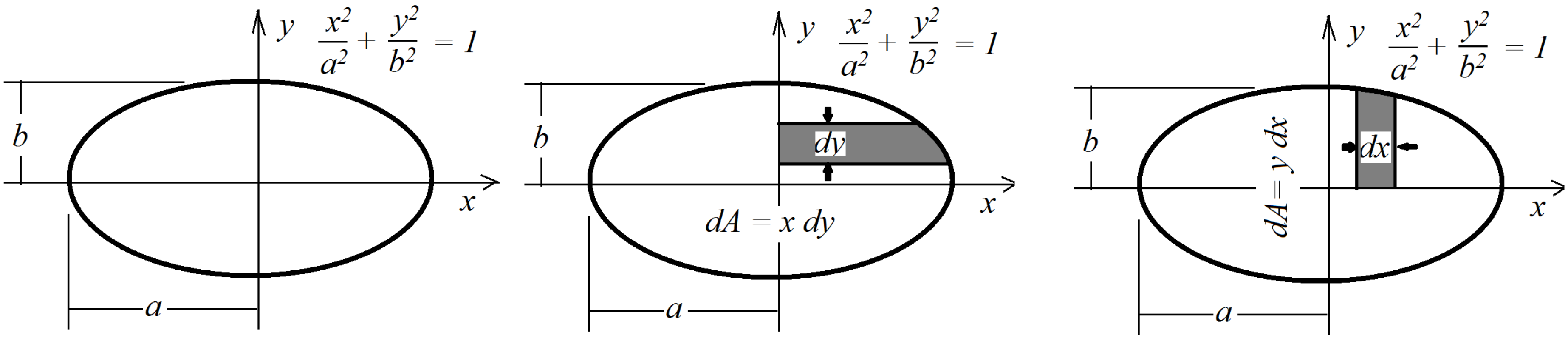


$$I_x = 2\int y^2\, dA \;=\; 2\int_0^h y^2\, \frac{b}{2h}\,(h - y)dy = \frac{1}{12}\, bh^3 \quad (unit)^4$$

$$I_y = 2\int x^2\, dA \;=\; 2\int_0^b x^2\,(h - \frac{2h}{b}\, x)\, dx = \frac{1}{48}\, b^3 h \quad (unit)^4$$

Problems (Moment of Inertia)

Example-2 (c): Find the Moment of Inertia for the area shown.

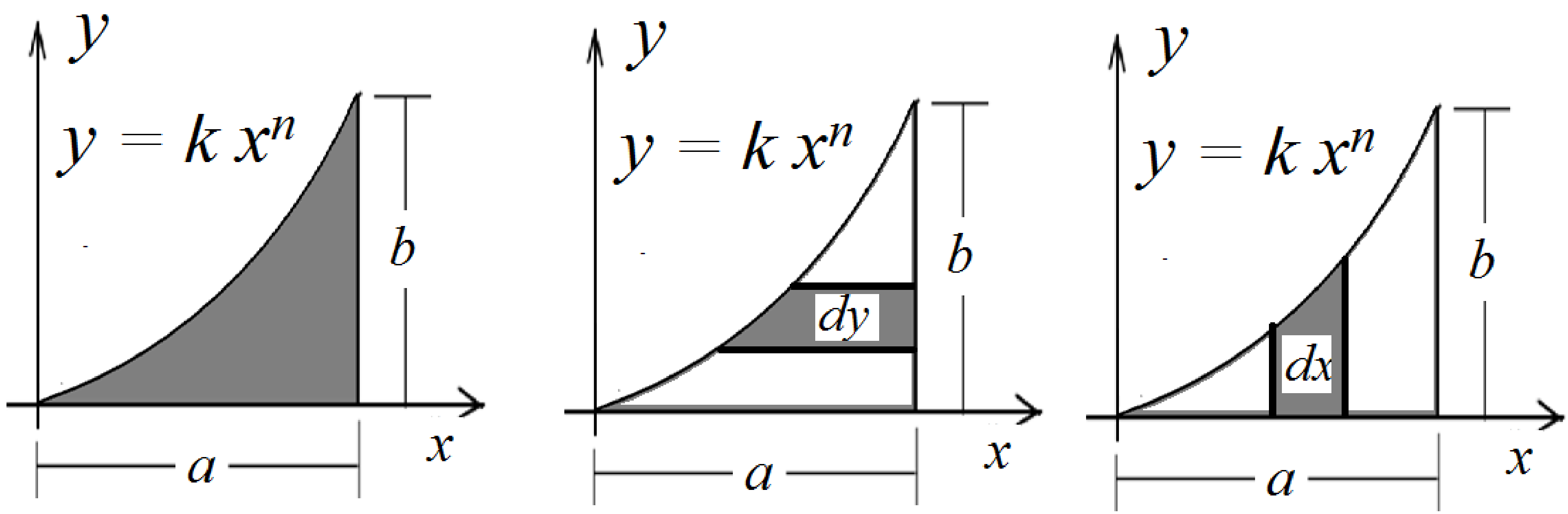


$$I_x = 2\int y^2\, dA = 2\int_{-b}^{b} y^2 \sqrt{a^2\left(1 - \frac{y^2}{b^2}\right)}\, dy = \frac{1}{4}\pi a b^3 \quad (unit)^4$$

$$I_y = 2\int x^2\, dA = 2\int_{-a}^{a} x^2 \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\, dx = \frac{1}{4}\pi a^3 b \quad (unit)^4$$

Problems (Moment of Inertia)

Example-2 (d): Find the Moment of Inertia for the area shown.

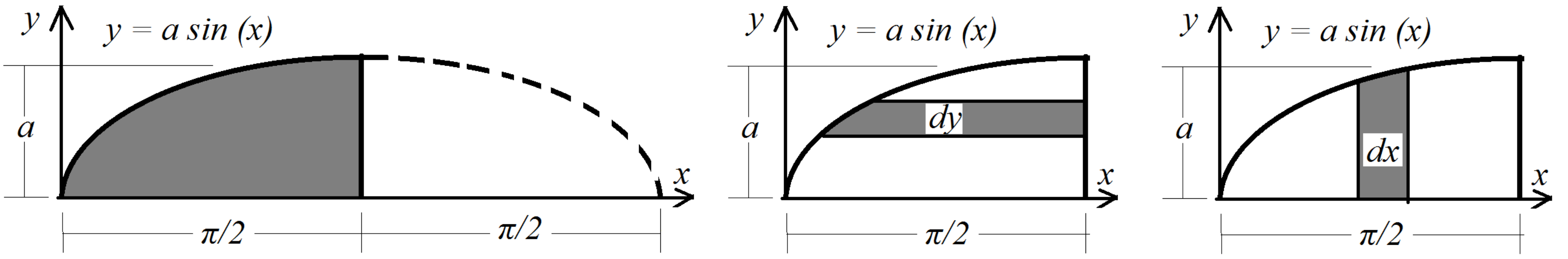


$$I_x = \int y^2\, dA = \int_0^b y^2\,(a - x)\,dy = \int_0^b y^2 \left(a - \left(\frac{y}{k}\right)^{\frac{1}{n}}\right) dy$$

$$I_y = \int x^2\, dA = \int_0^a x^2\, k\, x^n\, dx = \int_0^a k\, x^{n+2}\, dx$$

Moment of Inertia of an Area

Example-2 (e): Find the Moment of Inertia for the area shown.

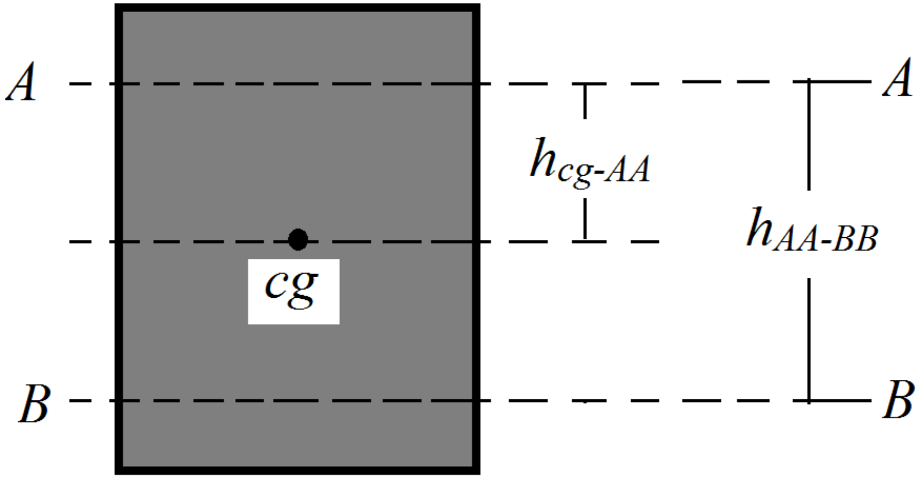


$$I_x = \int y^2\,dA = \int_0^a y^2\left(\frac{\pi}{2} - x\right)dy = \int_0^a y^2\left(\frac{\pi}{2} - \sin^{-1}\frac{y}{a}\right)dy$$

$$I_y = \int x^2\,dA = \int_0^{\frac{\pi}{2}} x^2\,\mathrm{a}\sin(x)\;dx = a\int_0^{\frac{\pi}{2}} x^2\;\sin(x)\;dx$$

Moment of Inertia

Parallel Axis Theorem: The relation between the moment of inertia about a line passing through the centroid and another parallel line is given as:



$$I_{AA} = I_{cg} + Ah^2_{cg-AA}$$

$$I_{AA} \neq I_{BB} + Ah^2_{BB-AA}$$

Always Remember:

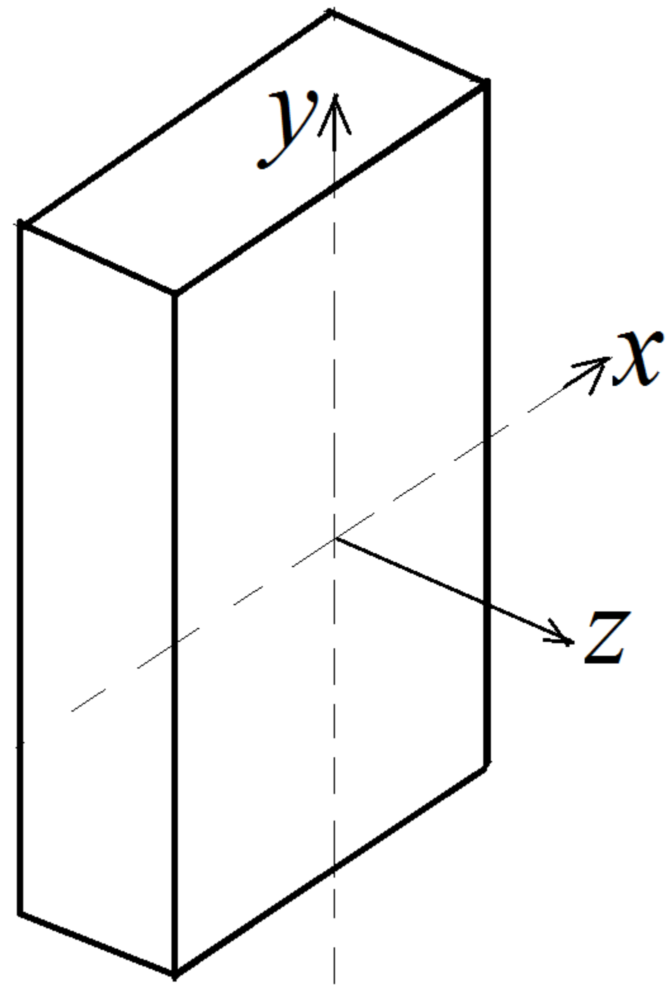
- Line passing through cg always yields minimum moment.
- To use parallel axis theorem one of the axes must be through centroid.

Polar moment of Inertia

- Sum of moment of inertia about both the axes in cartesian axes.

$$J = I_x + I_y$$

- Here *'J'* is the polar moment of inertia about the axis passing through the intersecting point of *x*-axis and *y*-axis and perpendicular to both of them.



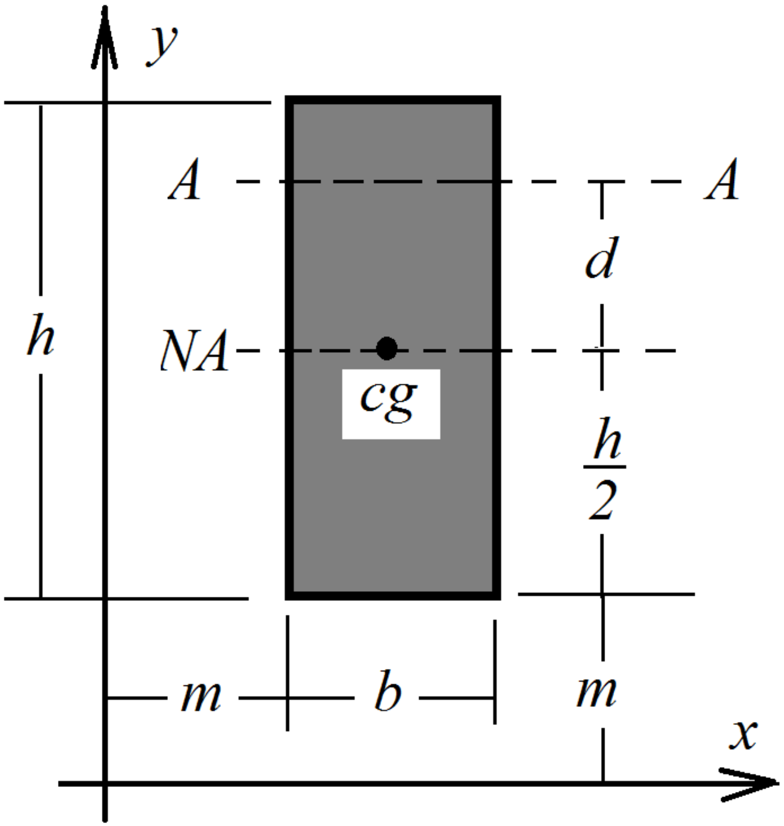
Problems (Moment of Inertia)

Example-3: Find the Moment of Inertia with respect to x-axis and line A-A for the area shown.

$$I_{NA} = \frac{1}{12} bh^3$$

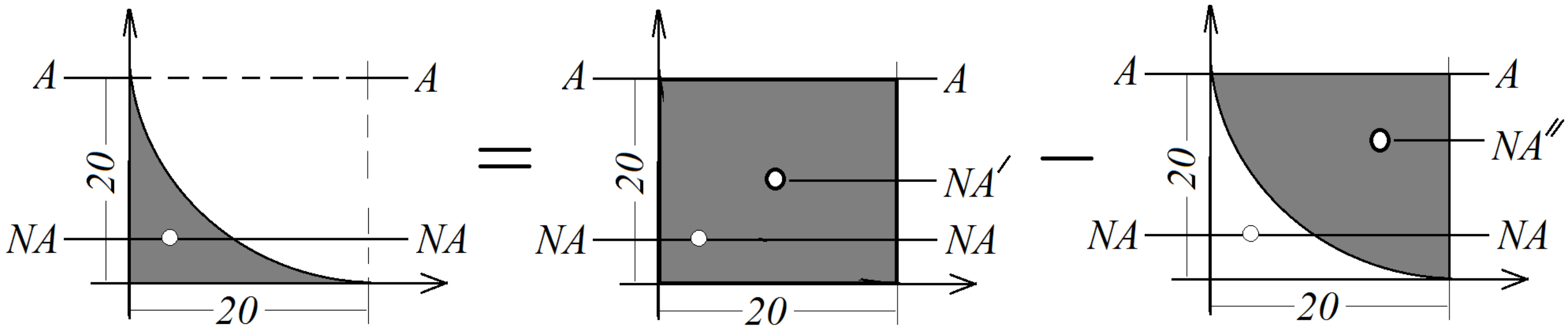
$$I_x = I_{NA} + A\, h_{cg-x}^2$$
$$= \frac{1}{12} bh^3 + bh\left(m + \frac{h}{2}\right)^2$$

$$I_{AA} = I_{NA} + A\, h_{cg-AA}^2$$
$$= \frac{1}{12} bh^3 + A(d)^2$$



Problems (Moment of Inertia)

Example-4: Find the Moment of Inertia for the area shown with respect to x-axis and line AA. (Also calculate I_{NA})

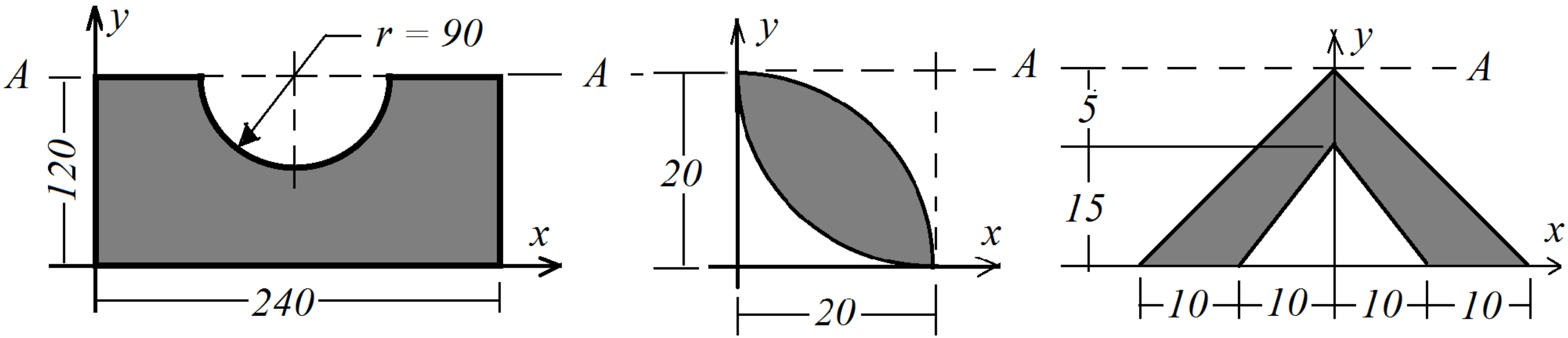


$\bar{x} = 4.51\ unit$ $\bar{y} = 4.51\ unit$

With respect to line A—A:	With respect to NA:	With respect to x-axis:
Rectangle: $I_{AA}^{R} = \frac{1}{3}\ 20 \times\ 20^3$	Rectangle: $I_{NA'}^{R} = I_{AA}^{R} -\ A\,(10)^2$ $I_{NA}^{R} = I_{NA'}^{R} +\ A\,(10-4.51)^2$	Rectangle: $I_{NA'}^{R} = I_{AA}^{R} -\ A\,(10)^2$ $I_{x}^{R} = I_{NA'}^{R} +\ A\,(10)^2$
Quarter Circle: $I_{AA}^{Q} = \frac{\pi}{16}\ 20^4$	Quarter Circle: $I_{NA''}^{Q} = I_{AA}^{Q} -\ A\,(8.49)^2$ $I_{NA}^{Q} = I_{NA''}^{Q} +\ A\,(11.51-4.51)^2$	Quarter Circle: $I_{NA''}^{Q} = I_{AA}^{Q} -\ A\,(8.49)^2$ $I_{x}^{Q} = I_{NA''}^{Q} +\ A\,(20-8.49)^2$
Overall: $I_{AA} = \ I_{AA}^{R} + I_{AA}^{Q}$	Overall: $I_{NA} = \ I_{NA}^{R} + I_{NA}^{Q}$	Overall: $I_{x} = \ I_{x}^{R} + I_{x}^{Q}$

Moment of Inertia

Assignment 4: Determine the moment of inertia for the areas shown with respect to NA, *x*-axis and line A—A.



Radius of Gyration

An imaginary concept defined as the distance from the reference axis so that if an area is placed and concentrated there it produces same moment of inertia.

$$I = A\,k^2$$

$$k = \sqrt{I/A}$$

Moment of Inertia of a Mass

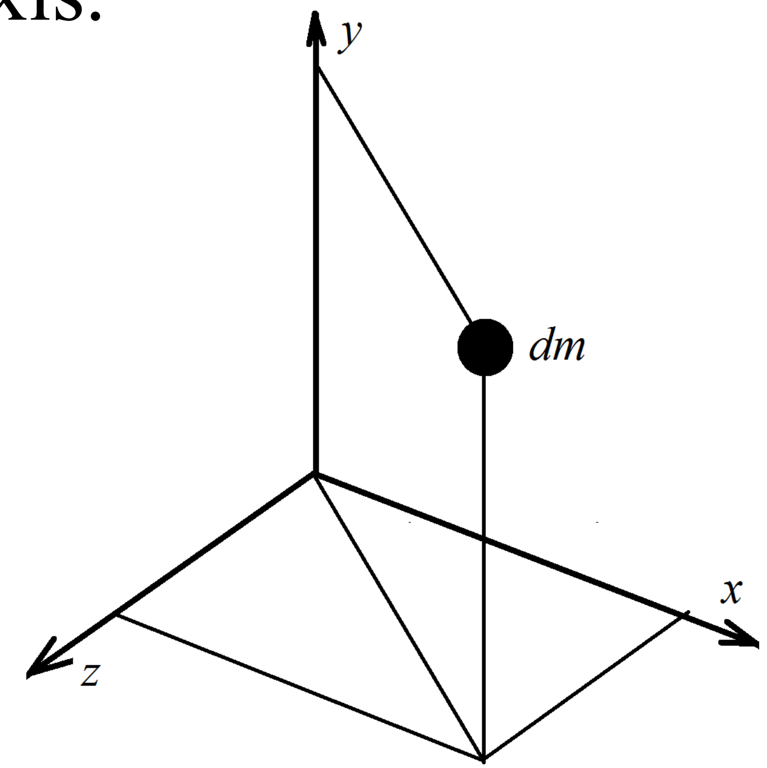
$$I = \int r^2 \, dm$$

Here *'r'* is the distance of the differential mass *'dm'* from the reference axis.

$$I_x = \int (y^2 + z^2) \, dm$$

$$I_y = \int (x^2 + z^2) \, dm$$

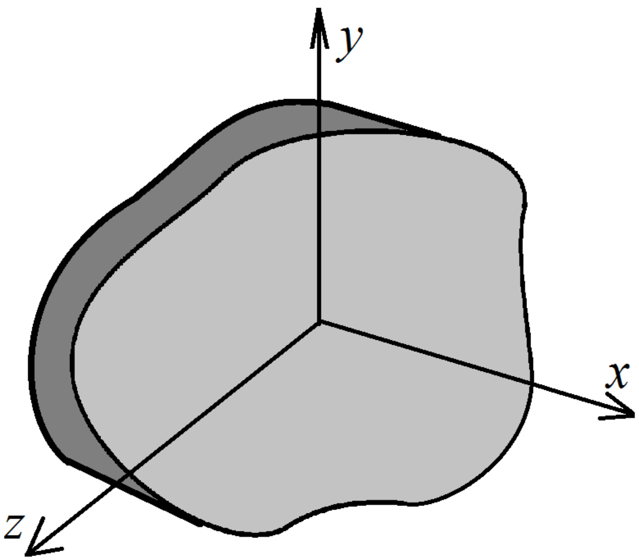
$$I_z = \int (x^2 + y^2) \, dm$$



Moment of Inertia for Thin Plate

$$I_x^m = \int r^2\, dm = \int r^2 \rho\, t\, dA = \rho\, t \int r^2\, dA = \rho\, t \times I_x^a$$

Here superscripts m and a represent mass and area respectively.



$I_x^m = Mass\ moment\ of\ Inertia\ with respect\ to\ x - axis$

$I_x^a = Area\ moment\ of\ Inertia\ with respect\ to\ x - axis$

Moment of Inertia for Thin Plate

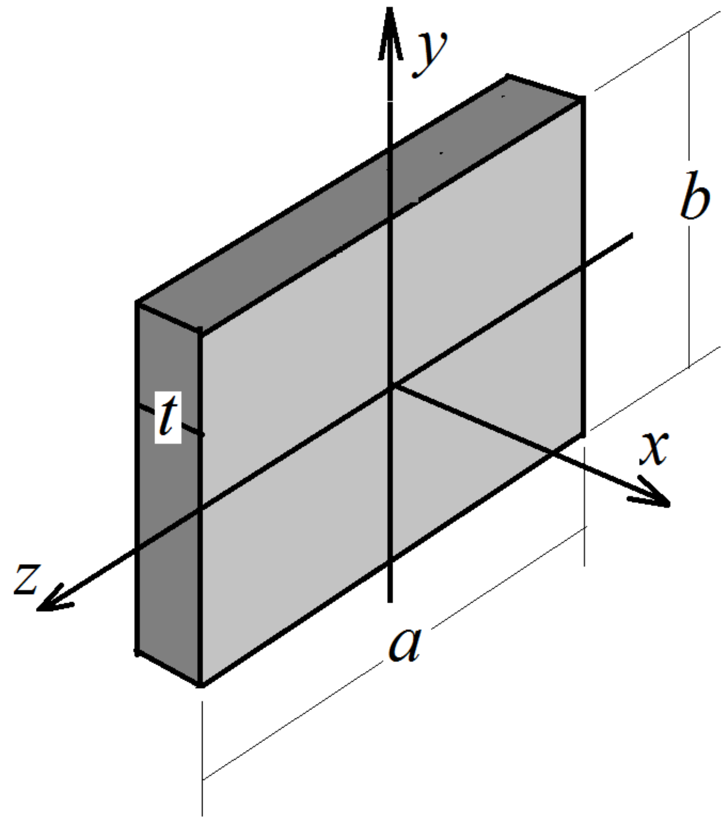
(Rectangular Plate)

$$I_x^a = \frac{1}{12}\,a^3 b + \frac{1}{12}\,ab^3$$

$$= \frac{1}{12}ab\,(a^2 + b^2)$$

$$I_x^m = \rho\, t \times I_x^a$$

$$= \rho\, t \times \frac{1}{12}ab\,(a^2 + b^2)$$

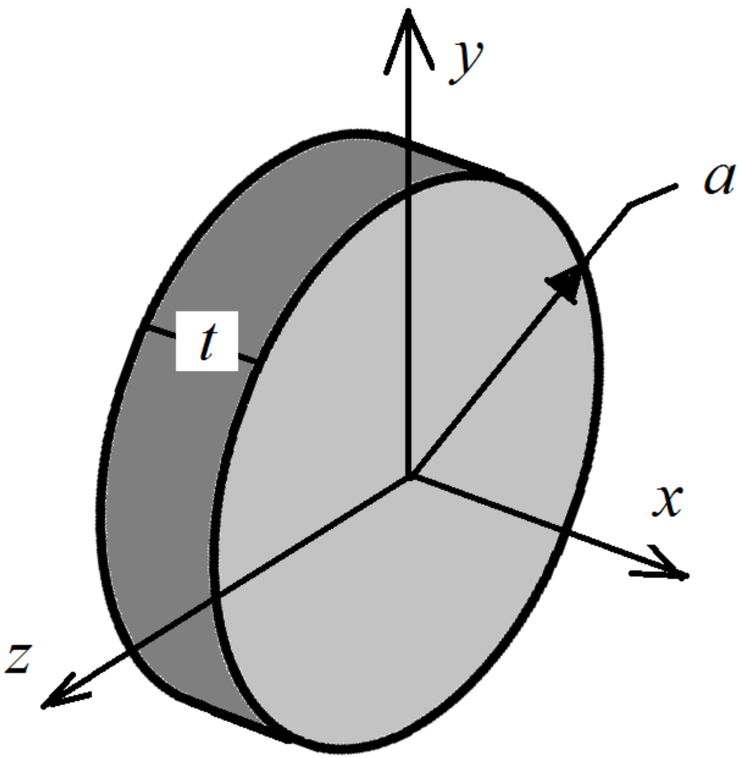


For elementary thin plate: ***(t = dx):*** $dI_x^m = \frac{1}{12}\,\rho \times ab(a^2 + b^2)\,dx$

Moment of Inertia for Thin Plate
(Circular Plate)

$$I_x^a = \frac{1}{4}\pi r^4 + \frac{1}{4}\pi r^4$$
$$= \frac{1}{2}\pi r^4$$

$$I_x^m = \rho\, t \times I_x^a$$
$$= \rho\, t \times \frac{1}{2}\pi r^4$$



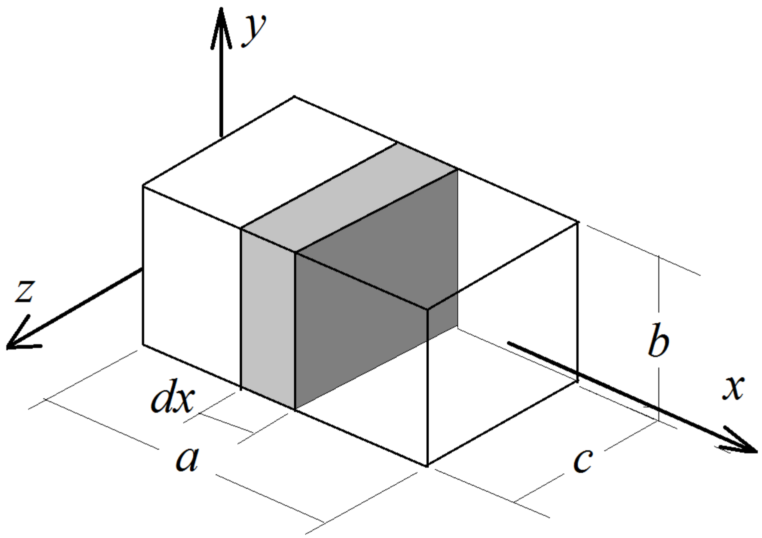
For elementary thin plate: *(t = dx)*: $dI_x^m = \frac{1}{2}\,\rho\,\pi r^4\, dx$

Problems (Moment of Inertia)

Example-5: Determine the moment of inertia for the solid shown in the figure. (about *x*, *y* and *z* axis)

$$dI_x = \frac{1}{12}\rho\, bc\,(b^2 + c^2)dx$$

$$I_x = \frac{1}{12}\rho\, bc\,(b^2 + c^2)\int_0^a dx$$



$$dm = \rho\, bc\, dx$$

$$dI_y = dI'_y + x^2 dm$$

$$I_y = \frac{1}{12}\rho\, bc^3\int_0^a dx + \rho\, bc\int_0^a x^2 dx$$

$$dI_z = dI'_z + x^2 dm$$

$$I_z = \frac{1}{12}\rho\, b^3 c\int_0^a dx + \rho\, bc\int_0^a x^2 dx$$

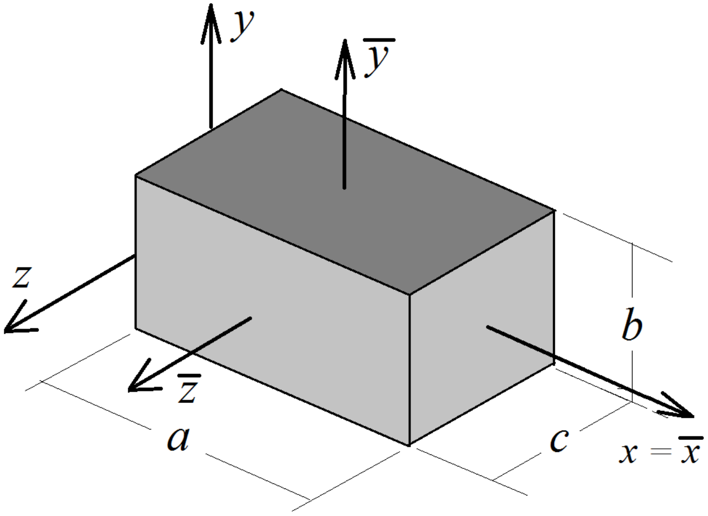
Problems (Moment of Inertia)

Example-6: Determine the moment of inertia for the solid shown in example 5 with respect to the axes passing through the centroid.

$$\bar{I}_x = I_x = \frac{1}{12}\, m\,(b^2 + c^2)$$

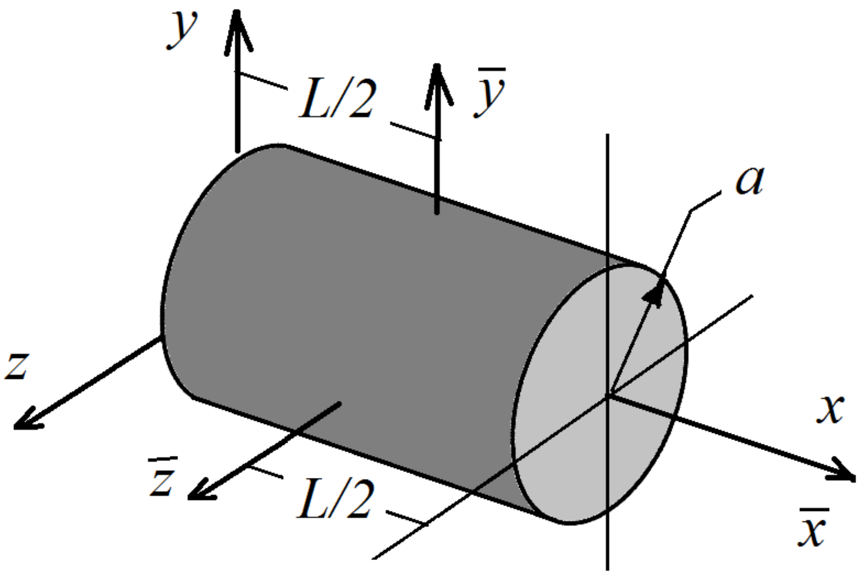
$$\bar{I}_y = I_y - \ m\,(\frac{a}{2})^2 = \frac{1}{12}\, m\,(a^2 + c^2)$$

$$\bar{I}_z = I_z - \ m\,(\frac{a}{2})^2 = \frac{1}{12}\, m\,(a^2 + b^2)$$



Problems (Moment of Inertia)

Example-7 (a): Determine the moment of inertia for the solid shown with respect to the axes and lines parallel to axes and passing through the centroid.



$$dm = \rho\pi\, a^2\, dx$$

$$dI_x = \frac{1}{2}\,\rho\,\pi a^4 dx$$

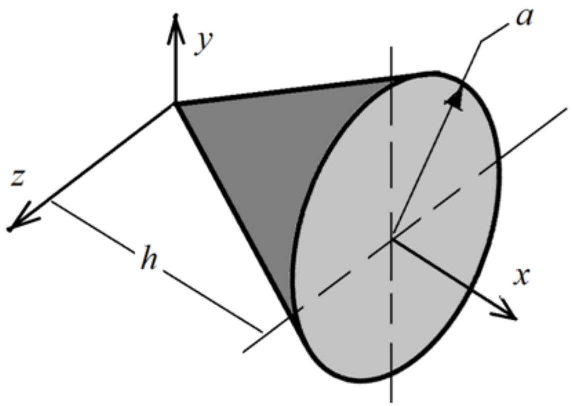
$$dI_y = dI_z = dI'_y + x^2\, dm$$

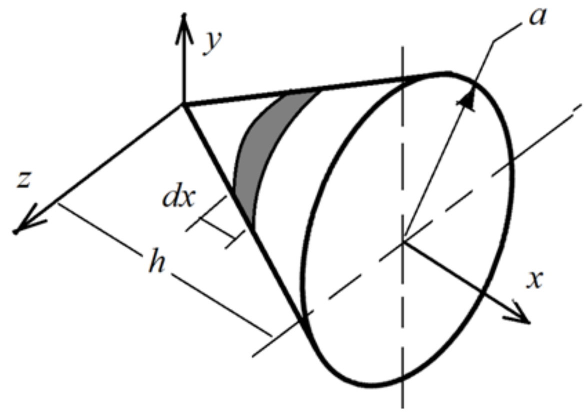
$$\bar{I}_x = I_x = \int_0^L \frac{1}{2}\,\rho\,\pi a^4\, dx$$

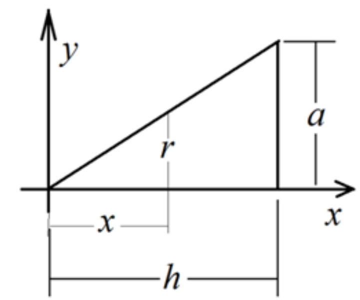
$$d\bar{I}_y = d\bar{I}_z = dI_y - m\,(\frac{L}{2})^2$$

Problems (Moment of Inertia)

Example-7(b): Determine the moment of inertia for the solid shown with respect to the axes and lines parallel to axes and passing through the centroid.







*From figure, r=(a/h)**

$$dI_x = \frac{\pi}{2}\,\rho\, r^4\, dx$$

$$dI_y = dI_y' + x^2 dm$$

$$I_x = \frac{\pi}{2}\,\rho \int_0^h \frac{a^4}{h^4}\, x^4 dx$$

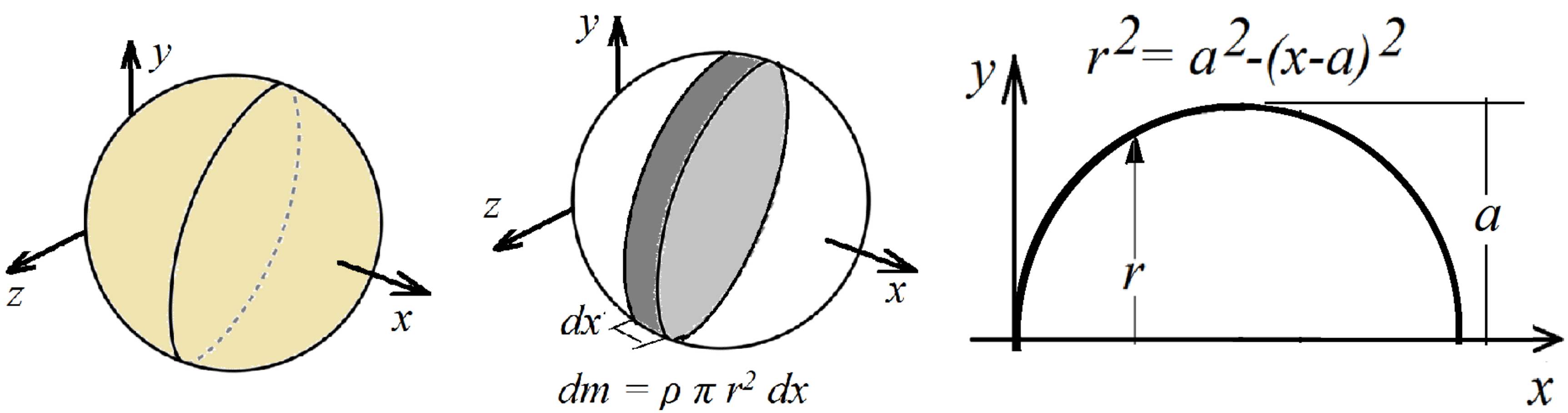
$$I_y = \frac{\pi}{4}\,\rho \int_0^h \frac{a^4}{h^4}\, x^4\, dx + \pi\,\rho \int_0^h \frac{a^2}{h^2}\, x^4 dx$$

$$\overline{I_x} = I_x = \frac{\pi}{2}\,\rho \int_0^h \frac{a^4}{h^4}\, x^4 dx$$

$$\overline{I_y} = I_y - m\,(\frac{3h}{4})^2$$

Problems (Moment of Inertia)

Example-8 (a): Determine the moment of inertia for the solid shown with respect to the axes and lines parallel to axes and passing through the centroid.



$dI_x = \frac{\pi}{2} \rho \, r^4 \, dx$	$I_x = \frac{\pi}{2} \rho \int_0^{2a} [a^2 - (x-a)^2]^2 dx$	$\bar{I}_x = I_x$

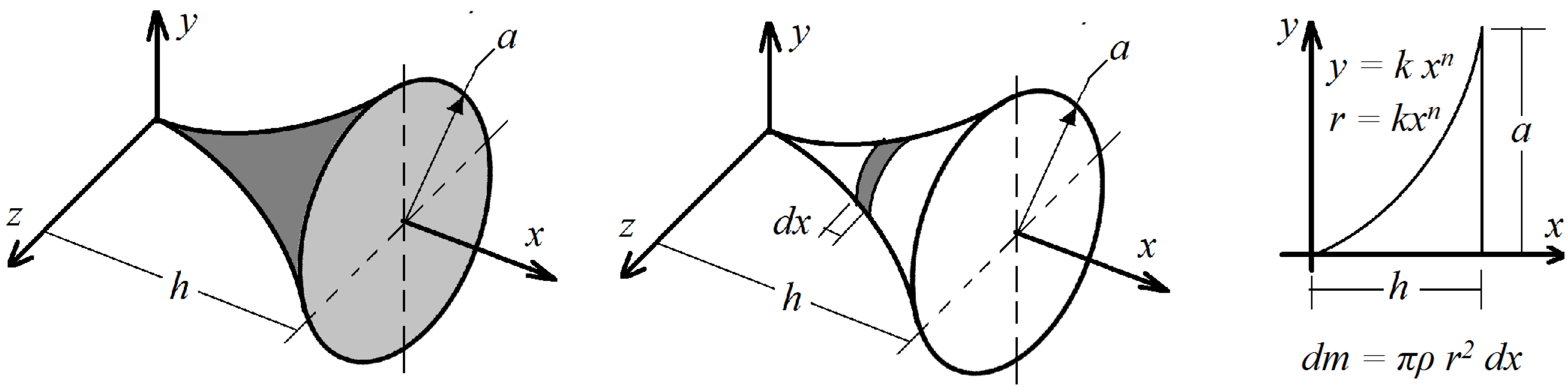
$$dI_y = dI'_y + x^2 dm$$

$$I_y = \frac{\pi}{4} \rho \int_0^{2a} [a^2 - (x-a)^2]^2 \, dx + \pi \, \rho \int_0^{h} x^2 [a^2 - (x-a)^2] dx$$

$$\bar{I}_y = I_y - m \, (a)^2$$

Problems (Moment of Inertia)

Example-8 (b): Determine the moment of inertia for the solid shown with respect to the axes and lines parallel to axes and passing through the centroid.



$dI_x = \frac{\pi}{2}\,\rho\, r^4\, dx$	$I_x = \frac{\pi}{2}\,\rho \int_0^h (kx^n)^4 dx$	$\overline{I_x} = I_x$

$$dI_y = dI'_y + \ x^2 dm$$

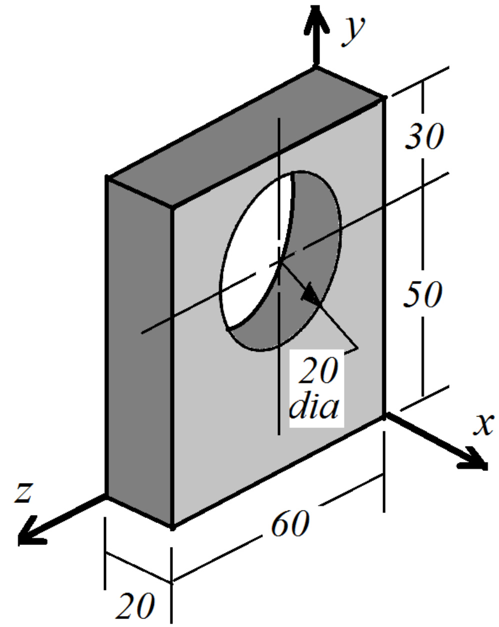
$$I_y = \ \frac{\pi}{4}\,\rho \int_0^h (kn^n)^4\, dx + \pi\,\rho \int_0^h x^2 (kx^n)^2]dx$$

$$\overline{I_y} = \ I_y - \ m\,(\tfrac{2n+1}{2n+2} \times h)^2$$

[From example-11 of Centroid Determination]

Problems (Moment of Inertia)

Example-9: Determine the moment of inertia for the object shown with respect to the axes of coordinate. Density of the material is 2000 kg/m^3. All dimensions are given in mm.



$$V^R = 96000\ mm^3\ ; \qquad m^R = 0.192\ kg$$

$$V^C = 25132.8\ mm^3\ ; \qquad m^C = 0.05\ kg$$

$$V = V^R - V^C = 70867.2\ mm^3$$

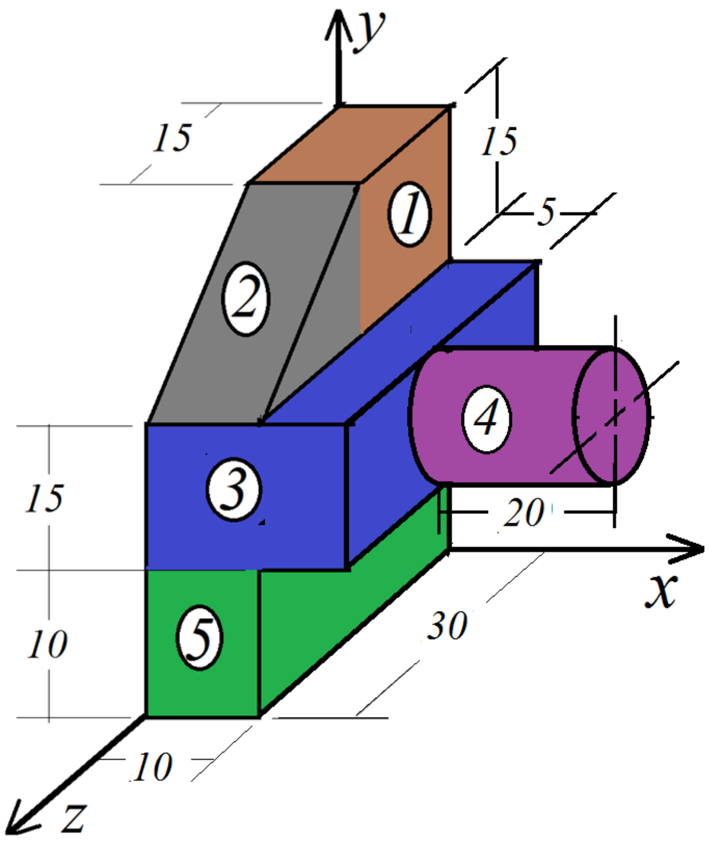
$$m = m^R - m^C = 0.142\ kg$$

$$I_{\acute{x}}^R = \frac{1}{12} m^R\,(60^2 + 80^2); \quad I_x^R = I_{\acute{x}}^R + m^R\,(30^2 + 40^2)$$

$$I_{\acute{x}}^C = \frac{1}{2} m^C\,(20)^2; \quad I_x^C = I_{\acute{x}}^C + m^C\,(30^2 + 50^2)$$

Problems (Moment of Inertia)

Assignment -5: Determine the moment of inertia for the object shown with respect to *x*-axis and *y*-axis.



ρ_1: 1000 kg/m^3

ρ_2: 800 kg/m^3

ρ_3: 1200 kg/m^3

ρ_4: 1500 kg/m^3

ρ_5: 2000 kg/m^3

?